A Proof of Melham's Identities

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Contents

1 Abstract

Melham [10] conjectures 21 identities, all of which are analogous to Jacobi's two-square theorem. Melham mentions that a small number of these have already been proved in various ways by Hirschhorn [5], Sun [13], and Dickson [3] (combined with work from Adiga, Cooper, and Han [1]). In this paper we offer a straightforward method to proving all of them.

2 Introduction

Fermat's two-square theorem says an odd prime, p, can be expressed as the sum of two squares if and only if p is congruent to 1 modulo 4. Jacobi expanded on this with Jacobi's two-square theorem, which tells us the number of distinct ways we can represent such a prime as the sum of two squares. Jacobi's theorem tells us the number of ways is four times the difference between the number of divisors of p congruent to 1 and the number of divisors congruent to 3 modulo 4 [6]. Instead of sums of squares, Melham's identities give the number of representations of a number (not just a prime) as the sum of multiples of triangular, pentagonal, or heptagonal numbers.

Define

$$G_3(q) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}, \qquad G_5(q) := \sum_{n=-\infty}^{\infty} q^{\frac{n(3n-1)}{2}}, \qquad G_7(q) := \sum_{n=-\infty}^{\infty} q^{\frac{n(5n-3)}{2}},$$

the generating functions of the triangular, pentagonal, and heptagonal numbers. Melham's identities (equations (6) to (25), which we refer to using the same numbers for simplicity, in his original paper [10]) are all of the form

$$G_k(q^\alpha)G_k(q^\beta) = \text{an explicit } q\text{-series}$$

where $k = 3, 5$ or 7, $\alpha, \beta \in \mathbb{N}$, and $q = e^{2\pi i\tau}$, with τ in the upper half plane, H. For example, his identity (6) is

$$G_3(q)G_3(q^5) = \sum_{n=0}^{\infty} \left[\frac{q^{3n} + q^{7n+1}}{1 - q^{20n+5}} - \frac{q^{13n+9} + q^{17n+12}}{1 - q^{20n+15}} \right].$$

The method we shall use relies on the theory of *modular forms.* We provide a brief description here, but for a more thorough reading, the reader should look to such sources as [2], [9], [11]. Other authors have attempted to prove these identities with alternative methods, from straight forward q-series manipulations to utilising binary quadratic forms. Alas, while some have been proven (the author believes around 6), the methods used have not been broad or powerful enough to prove more than a few. Our method is to show (in Section 3) that the LHS (raised to an even power) of each identity is a type of theta function, a modular form over a certain congruence subgroup. We then show in Section 4 that the RHS (raised to the same even power) is also a modular form for the same weight over this subgroup, by showing that it can be written in terms of the Weierstrass zeta function. It is then a simple case of checking a small number of coefficients to show that each identity satisfies Sturm's bound (described further in Section 7), and thus that both sides must be equivalent up to a constant, which we will show is 1.

A *congruence subgroup*, Γ, of $SL_2(\mathbb{Z})$ is a subgroup of $SL_2(\mathbb{Z})$ that contains

$$\Gamma(N) := \mathrm{Ker}\Big(SL_2(\mathbb{Z}) \to SL_2\left(\mathbb{Z}/N\mathbb{Z}\right)\Big)$$

where $\Gamma(N)$ is called the *principal congruence subgroup*. The smallest such N is the *level* of Γ. Three such particular congruence subgroups of interest are:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\},$$

$$\Gamma^0(N) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_0(N) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

where $*$ represents any number. All three groups have level N. Now define H to be the complex upper half plane. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, and for a complex function f, define

$$f|_A(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

A function f is a modular form of weight k, with k a positive integer, over a congruence subgroup Γ if f is holomorphic on H, $f|_\gamma$ remains bounded as $\Im(z) \to \infty$ for any $\gamma \in SL_2(\mathbb{Z})$, and verifies, for all $\gamma \in \Gamma$,

$$f|_\gamma = f.$$

3 The LHS as Theta Functions

We begin this section by defining a *lattice* in $\mathbb{R}^n$ as done in [8, p. 149]. A lattice in $\mathbb{R}^n$ is a subgroup of $\mathbb{R}^n$ of the form

$$\Gamma = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_n,$$

where $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set of vectors in $\mathbb{R}^n$. Take the inner product of two elements to be the normal dot product, and denote the inner product of an element x with itself as x^2. An *integral lattice* is a lattice where the inner product of any two elements in the lattice is integral. An *even lattice* is an integral lattice where the inner product of an element with itself (or norm) is always even. The *dual lattice*, denoted Λ^*, of a lattice Λ, is the lattice of vectors having integral inner products with all the elements of Λ. Define the *discriminant* of a lattice to be $\mathrm{disc}(\Lambda) = |\Lambda^*/\Lambda|$. Finally, the *level* of a lattice Λ is the minimum $N \in \mathbb{N}$ with Nx^2 even for all $x \in \Lambda^*$ [4, p. 91].

Ebeling [4, p. 86] defines a *generalised theta function* for a lattice $\Lambda \subset \mathbb{R}^n$, a point $z \in \mathbb{R}^n$, the variable $\tau \in H$, and a spherical polynomial P of degree r, as

$$\vartheta_{z+\Lambda,P}(\tau) := \sum_{x \in z+\Lambda} P(x) e^{\pi i \tau x^2} = \sum_{x \in z+\Lambda} P(x) q^{\frac{x^2}{2}}.$$

We have no need to discuss spherical polynomials, for our matters it is sufficient to state that the constant polynomial $P(x) = 1$ is spherical, of degree 0. We take $P(x) = 1$ from here on, and will drop it from our notation.

Lemma 3.1. *Let $\Lambda \subset \mathbb{R}^n$ (n even), be an even lattice of level N, $\rho \in \Lambda^*$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Then for odd $d > 0$, $c \neq 0$,*

$$\vartheta_{\rho+\Lambda}(A\tau) = (c\tau+d)^{\frac{n}{2}} \left(\frac{\Delta}{d}\right) e^{\pi i a b \rho^2} \vartheta_{a\rho+\Lambda}(\tau)$$

with $\Delta := (-1)^{\frac{n}{2}} \mathrm{disc}(\Lambda)$, $\left(\frac{\Delta}{d}\right)$ the Jacobi symbol, and for $c = 0$,

$$\vartheta_{\rho+\Lambda}(A\tau) = e^{\pi i a b \rho^2} \vartheta_{a\rho+\Lambda}(\tau).$$

Proof. The proof of this is Corollary 3.1 combined with the remarks before Theorem 3.2 of [4, pp. 92-94]. ∎

Now, as $\rho \in \Lambda^*$, $\rho^2 \in \mathbb{Q}$. Let $\rho^2 = \frac{u}{v}$, with $\gcd(u,v) = 1$. We have

$$\vartheta^{2v}_{\rho+\Lambda}|_A = e^{2\pi iabu}\vartheta^{2v}_{a\rho+\Lambda} = \vartheta^{2v}_{a\rho+\Lambda}.$$

We now take $n = 2$, and consider lattices in $\mathbb{R}^2$.

3.1 Triangular Numbers, $G_3(q)$

Let $\Lambda = \left\langle \begin{pmatrix} \sqrt{2\alpha} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2\beta} \end{pmatrix} \right\rangle$, with α, β integers. Then Λ is even, and has dual

$$\Lambda^* = \left\langle \begin{pmatrix} \frac{1}{\sqrt{2\alpha}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2\beta}} \end{pmatrix} \right\rangle.$$

Calculating the level N, of Λ is straightforward, and gives $N = 4 \cdot \mathrm{lcm}(\alpha, \beta)$. Let $\rho = \begin{pmatrix} \sqrt{\frac{\alpha}{2}} \\ \sqrt{\frac{\beta}{2}} \end{pmatrix} \in \Lambda^*$. We have, with $q = e^{2\pi i\tau}$,

$$\begin{aligned}\vartheta_{\rho+\Lambda}(\tau) &= \sum_{n,m\in\mathbb{Z}} q^{\alpha n^2+\alpha n+\frac{\alpha}{4}+\beta m^2+\beta m+\frac{\beta}{4}} \\ &= q^{\frac{\alpha+\beta}{4}} \sum_{n,m\in\mathbb{Z}} q^{2\alpha\frac{n(n+1)}{2}+2\beta\frac{m(m+1)}{2}}.\end{aligned}$$

Hence

$$\vartheta_{\rho+\Lambda}\left(\frac{\tau}{2}\right) = q^{\frac{\alpha+\beta}{8}} \sum_{n\in\mathbb{Z}} q^{\alpha\frac{n(n+1)}{2}} \sum_{m\in\mathbb{Z}} q^{\beta\frac{m(m+1)}{2}},$$

so

$$\frac{1}{4}q^{-\frac{\alpha+\beta}{8}}\vartheta_{\rho+\Lambda}\left(\frac{\tau}{2}\right) = G_3(q^\alpha)G_3(q^\beta).$$

Now, it is easy to see that $\rho + \Lambda = a\rho + \Lambda$ whenever $a \equiv 1 \pmod 2$. Further, as N is clearly even, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we can see that a must be odd. Now, $\rho^2 = \frac{\alpha+\beta}{2}$, hence $\vartheta_{\rho+\Lambda}(\tau)^{2v}$ is a weight $2v$ modular form for $\Gamma_0(N)$ where $v = 1$ if $\alpha+\beta$ is even, and $v = 2$ if $\alpha+\beta$ is odd. Therefore $\vartheta_{\rho+\Lambda}(\frac{\tau}{2})^{2v}$ is a weight $2v$ modular form for $\Gamma' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We see for $A \in \Gamma'$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $b \equiv 0 \pmod 2$, and $c \equiv 0 \pmod{\frac{N}{2}}$. In other words,

$$\Gamma' = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0\left(\frac{N}{2}\right) \quad : \quad b \text{ even} \right\} = \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2).$$

Hence we can conclude that

$$\left(q^{\frac{\alpha+\beta}{8}} G_3(q^\alpha)G_3(q^\beta)\right)^{2v}$$

is a modular form for Γ' as well.

3.2 Pentagonal Numbers, $G_5(q)$

Let $\Lambda = \left\langle \begin{pmatrix} \sqrt{6\alpha} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{6\beta} \end{pmatrix} \right\rangle$, with α, β integers. Then Λ is even, and has dual

$$\Lambda^* = \left\langle \begin{pmatrix} \frac{1}{\sqrt{6\alpha}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{6\beta}} \end{pmatrix} \right\rangle.$$

The level of Λ is $N = 12 \cdot \mathrm{lcm}(\alpha, \beta)$. Let $\rho = \begin{pmatrix} -\sqrt{\frac{\alpha}{6}} \\ -\sqrt{\frac{\beta}{6}} \end{pmatrix} \in \Lambda^*$. We find

$$q^{-\frac{\alpha+\beta}{24}} \vartheta_{\rho+\Lambda}\left(\frac{\tau}{2}\right) = G_5(q^\alpha) G_5(q^\beta).$$

Again, it is easy to see that $\rho + \Lambda = a\rho + \Lambda$ whenever $a \equiv 1 \pmod 6$. When $a \equiv -1 \pmod 6$ we see $a\rho + \Lambda = -(\rho + \Lambda)$, but as we are taking the norm of each shifted lattice point this is acceptable also. Further, as N is clearly even, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we can see that a must be odd and coprime to 3.

Now, $\rho^2 = -\frac{\alpha+\beta}{6}$, hence $\vartheta_{\rho+\Lambda}(\tau)^{2v}$ is a weight $2v$ modular form for the congruence subgroup $\Gamma' = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \quad : \quad a \equiv \pm 1 \pmod 6\}$ where

$$v = \begin{cases} 1 & \alpha + \beta \equiv 0 \pmod 6 \\ 6 & \alpha + \beta \equiv \pm 1 \pmod 6 \\ 3 & \alpha + \beta \equiv \pm 2 \pmod 6 \\ 2 & \alpha + \beta \equiv 3 \pmod 6. \end{cases}$$

Therefore $\vartheta_{\rho+\Lambda}(\frac{\tau}{2})^{2v}$ is a weight $2v$ modular form for $\Gamma'' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Gamma' \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We see for $A \in \Gamma''$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $b \equiv 0 \pmod 2$, and $c \equiv 0 \pmod{\frac{N}{2}}$. Recalling that $12 \nmid N$, we therefore have,

$$\Gamma'' = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0\left(\frac{N}{2}\right) \quad : \quad b \text{ even}, \ a \equiv \pm 1 \pmod 6 \right\} = \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2)$$

as $1 = ad - bc \equiv d \pmod 6$. Hence we can conclude that

$$\left(q^{\frac{\alpha+\beta}{24}} G_5(q^\alpha) G_5(q^\beta)\right)^{2v}$$

is a modular form for Γ'' as well.

3.3 Heptagonal Numbers, $G_7(q)$

Let $\Lambda = \left\langle \begin{pmatrix} \sqrt{10\alpha} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{10\beta} \end{pmatrix} \right\rangle$, with α, β integers. Then Λ is even, and has dual

$$\Lambda^* = \left\langle \begin{pmatrix} \frac{1}{\sqrt{10\alpha}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{10\beta}} \end{pmatrix} \right\rangle.$$

The level of Λ is $N = 20 \cdot \mathrm{lcm}(\alpha, \beta)$. Let $\rho = \begin{pmatrix} -3\sqrt{\frac{\alpha}{10}} \\ -3\sqrt{\frac{\beta}{10}} \end{pmatrix} \in \Lambda^*$. We find

$$q^{-\frac{9(\alpha+\beta)}{40}} \vartheta_{\rho+\Lambda}\left(\frac{\tau}{2}\right) = G_7(q^\alpha) G_7(q^\beta).$$

Once more, we have $\rho + \Lambda = a\rho + \Lambda$ whenever $a \equiv 1 \pmod{10}$. Further, as N is clearly even, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we can see that a must be odd.

Now, $\rho^2 = \frac{\alpha+\beta}{10}$, hence $\vartheta_{\rho+\Lambda}(\tau)^{2v}$ is a weight $2v$ modular form for the congruence subgroup $\Gamma' = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \quad : \quad a \equiv 1 \pmod{10}\}$ where

$$v = \begin{cases} 1 & \alpha+\beta \equiv 0 \pmod{10} \\ 10 & \alpha+\beta \equiv \pm 1 \pmod{10} \\ 5 & \alpha+\beta \equiv \pm 2 \pmod{10} \\ 10 & \alpha+\beta \equiv \pm 3 \pmod{10} \\ 5 & \alpha+\beta \equiv \pm 4 \pmod{10} \\ 2 & \alpha+\beta \equiv 5 \pmod{10}. \end{cases}$$

Therefore $\vartheta_{\rho+\Lambda}(\frac{\tau}{2})^{2v}$ is a weight $2v$ modular form for $\Gamma'' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Gamma' \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We see for $A \in \Gamma''$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $b \equiv 0 \pmod 2$, and $c \equiv 0 \pmod{\frac{N}{2}}$. In other words,

$$\Gamma'' = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0\left(\frac{N}{2}\right) \quad : \quad b \text{ even, } a \equiv \pm 1 \pmod{10} \right\} = \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2) \cap \Gamma_1(10).$$

4 The RHS as the Weierstrass Zeta Function

The *Weierstrass zeta function* is a function that naturally leads to an Eisenstein series of weight 1 [2, p. 138]. We write it as Z to avoid confusion, and define it as

$$Z_\Lambda(z) := \frac{1}{z} + \sum_{\substack{w \in \Lambda \\ w \neq 0 \\ w \neq z}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right)$$

for a 2 dimensional lattice $\Lambda := \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$, and $z \in \mathbb{C}$. This function is not quite periodic, but rather with each step along the lattice increases by a lattice constant, $\eta_1(\Lambda)$ or $\eta_2(\Lambda)$, defined by:

$$\eta_1(\Lambda) := Z_\Lambda(z+\omega_1) - Z_\Lambda(z) \quad \text{and} \quad \eta_2(\Lambda) := Z_\Lambda(z+\omega_2) - Z_\Lambda(z).$$

If we assume without loss of generality that $\Im\left(\frac{\omega_1}{\omega_2}\right) > 0$ then these lattice constants satisfy the *Legendre relation*,

$$\eta_2(\Lambda)\omega_1 - \eta_1(\Lambda)\omega_2 = 2\pi i.$$

We can thus define a periodic function, $Z^*_{\Lambda_\tau}$, for $u, v \in \mathbb{R}$, and $\Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z}$, as follows,

$$Z^*_{\Lambda_\tau}(u\tau + v) := Z_{\Lambda_\tau}(u\tau + v) - u\eta_1(\Lambda_\tau) - v\eta_2(\Lambda_\tau).$$

For some scalar m, and writing $Z(z \mid \omega_1, \omega_2) := Z_\Lambda(z)$ as is common, we see

$$Z(mz \mid m\omega_1, m\omega_2) = \frac{1}{mz} + \sum_{\substack{w \in m\Lambda \\ w \neq 0 \\ w \neq mz}} \left(\frac{1}{mz - w} + \frac{1}{w} + \frac{mz}{w^2} \right) = \frac{1}{m} \left(\frac{1}{z} + \sum_{\substack{w \in \Lambda \\ w \neq 0 \\ w \neq z}} \left(\frac{1}{z - w} + \frac{1}{w} + \frac{z}{w^2} \right) \right)$$

$$= \frac{1}{m} Z(z \mid \omega_1, \omega_2).$$

Applying this to $Z(z \mid 1, \tau) = Z_{\Lambda_\tau}(z)$ transformed by a matrix in $SL_2(\mathbb{Z})$ we find

$$Z\left(z \mid 1, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d) Z(z(c\tau + d) \mid c\tau + d, a\tau + b)$$

$$= (c\tau + d) Z(z(c\tau + d) \mid 1, \tau)$$

as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. We define

$$f_{r,s}(\tau) := Z\left(\frac{r\tau + s}{N} \mid 1, \tau\right).$$

Hence

$$f_{r,s}\left(\frac{a\tau + b}{c\tau + d}\right) = Z\left(\frac{1}{N}\left(s + r\left(\frac{a\tau + b}{c\tau + d}\right)\right) \mid 1, \frac{a\tau + b}{c\tau + d}\right)$$

$$= (c\tau + d) Z\left(\frac{1}{N}\left(s(c\tau + d) + r(a\tau + b)\right) \mid 1, \tau\right)$$

$$= (c\tau + d) f_{ar+cs, br+ds}(\tau).$$

Diamond and Shurman [2, p. 138] further state that the Weierstrass zeta function, with $q := e^{2\pi i \tau}$, can be expressed as

$$Z_{\Lambda_\tau}(z) = \eta_2(\Lambda_\tau) z - \pi i \frac{1 + e^{2\pi i z}}{1 - e^{2\pi i z}} - 2\pi i \sum_{n=1}^{\infty} \left[\frac{e^{2\pi i z} q^n}{1 - e^{2\pi i z} q^n} - \frac{e^{-2\pi i z} q^n}{1 - e^{-2\pi i z} q^n} \right].$$

We consider the holomorphic function of τ,

$$Z^*_{\Lambda_\tau}\left(\frac{a\tau + b}{N}\right) = Z_{\Lambda_\tau}\left(\frac{a\tau + b}{N}\right) - \frac{a\eta_1(\Lambda_\tau) + b\eta_2(\Lambda_\tau)}{N}$$

which we call an Eisenstein series of weight 1.

4.1 $E_{a,b,N}$ when $a \not\equiv 0 \pmod{N}$

If we have $a \not\equiv 0(N)$ we can expand the denominators, so we take $0 < a < N$ and using the fact that $e^{2\pi i z} = e^{2\pi i \left(\frac{a\tau + b}{N}\right)} = e^{2\pi i \frac{a\tau}{N}} e^{2\pi i \frac{b}{N}} = q^{\frac{a}{N}} \zeta_N^b$ (with $\zeta_N := e^{\frac{2\pi i}{N}}$ as always), simplify, using the Legendre relation:

$$Z^*_{\Lambda_\tau}\left(\frac{a\tau + b}{N}\right) = \eta_2(\Lambda_\tau)\left(\frac{a\tau + b}{N}\right) - \pi i \frac{1 + q^{\frac{a}{N}} \zeta_N^b}{1 - q^{\frac{a}{N}} \zeta_N^b} - 2\pi i \sum_{n=1}^{\infty} \left[\frac{q^{\frac{a}{N}} \zeta_N^b q^n}{1 - q^{\frac{a}{N}} \zeta_N^b q^n} - \frac{q^{-\frac{a}{N}} \zeta_N^{-b} q^n}{1 - q^{-\frac{a}{N}} \zeta_N^{-b} q^n} \right]$$
$$- \frac{a\eta_1(\Lambda_\tau) + b\eta_2(\Lambda_\tau)}{N}$$

$$= \eta_2(\Lambda_\tau)\left(\frac{b}{N} - \frac{b}{N}\right) + \frac{a}{N}\left(\tau\eta_2(\Lambda_\tau) - \eta_1(\Lambda_\tau)\right) - \pi i\left(1 + 2\sum_{m=1}^{\infty} q^{\frac{am}{N}}\zeta_N^{bm}\right)$$

$$-2\pi i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[q^{\frac{m(a+nN)}{N}}\zeta_N^{bm} - q^{\frac{m(-a+nN)}{N}}\zeta_N^{-bm}\right]$$

$$= \frac{2\pi i a}{N} - \pi i - 2\pi i\sum_{m=1}^{\infty} q^{\frac{am}{N}}\zeta_N^{bm} - 2\pi i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[q^{m(\frac{a}{N}+n)}\zeta_N^{bm} - q^{\frac{m(-a+nN)}{N}}\zeta_N^{-bm}\right]$$

$$= 2\pi i\left(\frac{a}{N} - \frac{1}{2} - \sum_{m=1}^{\infty}\left[\sum_{n=0}^{\infty} q^{\frac{m}{N}(a+nN)}\zeta_N^{bm} - \sum_{n=1}^{\infty} q^{\frac{m}{N}(-a+nN)}\zeta_N^{-bm}\right]\right)$$

$$= 2\pi i\left(\frac{a}{N} - \frac{1}{2} - \sum_{r=1}^{\infty} q^{\frac{r}{N}}\left[\sum_{\substack{s\mid r\\ s\equiv a\ (N)\\ s>0}} \zeta_N^{\frac{br}{s}} - \sum_{\substack{s\mid r\\ s\equiv -a\ (N)\\ s>0}} \zeta_N^{-\frac{br}{s}}\right]\right) := E_{a,b,N}(\tau).$$

Hence we have replaced $Z^*_{\Lambda_\tau}$ with $E_{a,b,N}$, to make our choices of a, b, N explicit. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then by our earlier result involving the transformation of $f_{r,s}(\tau)$,

$$E_{\alpha,\beta,N}|_M = E_{a\alpha+c\beta,b\alpha+d\beta,N}.$$

The first two indices can be reduced modulo N, hence we see that $E_{a,b,N}$ remains invariant under transformation by $\Gamma(N)$.

4.2 $E_{a,b,N}$ when $a \equiv 0 \pmod N$

For the case $a \equiv 0 \pmod N$ we start with

$$Z^* = \eta_2(\Lambda)\left(\frac{a\tau + b}{N}\right) - \pi i\frac{1 + q^{\frac{a}{N}}\zeta_N^b}{1 - q^{\frac{a}{N}}\zeta_N^b} - 2\pi i\sum_{n=1}^{\infty}\left[\frac{q^{\frac{a}{N}}\zeta_N^b q^n}{1 - q^{\frac{a}{N}}\zeta_N^b q^n} - \frac{q^{-\frac{a}{N}}\zeta_N^{-b}q^n}{1 - q^{-\frac{a}{N}}\zeta_N^{-b}q^n}\right] - \frac{a\eta_1(\Lambda) + b\eta_2(\Lambda)}{N}$$

we have $a = tN$, with $t \in \mathbb{Z}$. We consider $t = 0$, i.e. $a = 0$, and notice that the result will hold for any $a \equiv 0 \pmod N$ due to periodicity.

$$Z^* = \eta_2(\Lambda)\left(\frac{b-b}{N}\right) - \pi i\frac{1 + \zeta_N^b}{1 - \zeta_N^b} - 2\pi i\sum_{n=1}^{\infty}\left[\frac{\zeta_N^b q^n}{1 - \zeta_N^b q^n} - \frac{\zeta_N^{-b}q^n}{1 - \zeta_N^{-b}q^n}\right]$$

$$= -\pi i\frac{1 + \zeta_N^b}{1 - \zeta_N^b} - 2\pi i\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\zeta_N^{mb}q^{mn} - \zeta_N^{-mb}q^{mn}\right]$$

$$= 2\pi i\left(-\frac{1}{2}\cdot\frac{1 + \zeta_N^b}{1 - \zeta_N^b} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{nm}\left[\zeta_N^{mb} - \zeta_N^{-mb}\right]\right)$$

$$= 2\pi i\left(\frac{1}{2} + \frac{1}{\zeta_N^b - 1} - \sum_{t=1}^{\infty}\sum_{\substack{s\mid t\\ s>0}} q^t\left[\zeta_N^{\frac{bt}{s}} - \zeta_N^{-\frac{bt}{s}}\right]\right)$$

$$= 2\pi i\left(\frac{1}{2} + \frac{1}{\zeta_N^b - 1} - \sum_{r=1}^{\infty}\sum_{\substack{s\mid r\\ s\equiv 0\ (N)\\ s>0}} q^{\frac{r}{N}}\left[\zeta_N^{\frac{br}{s}} - \zeta_N^{-\frac{br}{s}}\right]\right) := E_{0,b,N}(\tau).$$

4.3 Defining $F_{a,c,N}$ for $a, c \not\equiv 0(N)$

Now we consider the following linear combination, for $0 < a < N$,

$$\frac{1}{2\pi i}F_{a,c,N}(\tau) := \frac{1}{2\pi i}\sum_{b=0}^{N-1}\zeta_N^{-bc}E_{a,b,N}(\tau)$$

$$= \sum_{b=0}^{N-1}\zeta_N^{-bc}\left(\frac{a}{N}-\frac{1}{2}\right) - \sum_{b=0}^{N-1}\zeta_N^{-bc}\sum_{r=1}^{\infty}\left[\sum_{\substack{s|r\\ s\equiv a\ (N)\\ s>0}} q^{\frac{r}{N}}\zeta_N^{\frac{br}{s}} - \sum_{\substack{s|r\\ s\equiv -a\ (N)\\ s>0}} q^{\frac{r}{N}}\zeta_N^{\frac{-br}{s}}\right]$$

$$= \left(\frac{a}{N}-\frac{1}{2}\right)\sum_{b=0}^{N-1}\zeta_N^{-bc} - \sum_{r=1}^{\infty}\left[\sum_{\substack{s|r\\ s\equiv a\ (N)\\ s>0}} q^{\frac{r}{N}}\sum_{b=0}^{N-1}\zeta_N^{b\left(\frac{r}{s}-c\right)} - \sum_{\substack{s|r\\ s\equiv -a\ (N)\\ s>0}} q^{\frac{r}{N}}\sum_{b=0}^{N-1}\zeta_N^{b\left(-\frac{r}{s}-c\right)}\right]$$

$$= \left(\frac{a}{N}-\frac{1}{2}\right)\sum_{b=0}^{N-1}\zeta_N^{-bc} - \sum_{r=1}^{\infty}\left[\sum_{\substack{s|r\\ s\equiv a\ (N)\\ \frac{r}{s}\equiv c\ (N)\\ s>0}} q^{\frac{r}{N}}N - \sum_{\substack{s|r\\ s\equiv -a\ (N)\\ \frac{r}{s}\equiv -c\ (N)\\ s>0}} q^{\frac{r}{N}}N\right]$$

$$= \left(\frac{a}{N}-\frac{1}{2}\right)\sum_{b=0}^{N-1}\zeta_N^{-bc} - N\left(\sum_{\substack{s\equiv a\ (N)\\ t\equiv c\ (N)\\ s,t>0}} q^{\frac{st}{N}} - \sum_{\substack{s\equiv -a\ (N)\\ t\equiv -c\ (N)\\ s,t>0}} q^{\frac{st}{N}}\right).$$

So if we pick c such that $N \nmid c$, the sum on the left disappears, and we have

$$F_{a,c,N}(\tau) = -2\pi i N\left(\sum_{\substack{s\equiv a\ (N)\\ t\equiv c\ (N)\\ s,t>0}} q^{\frac{st}{N}} - \sum_{\substack{s\equiv -a\ (N)\\ t\equiv -c\ (N)\\ s,t>0}} q^{\frac{st}{N}}\right).$$

For simplicity we define

$$F^*_{a,c,N}(\tau) := \sum_{\substack{s\equiv a\ (N)\\ t\equiv c\ (N)\\ s,t>0}} q^{\frac{st}{N}} - \sum_{\substack{s\equiv -a\ (N)\\ t\equiv -c\ (N)\\ s,t>0}} q^{\frac{st}{N}} = -\frac{1}{2\pi i N}F_{a,c,N}(\tau).$$

We write

$$K_{a,c,N} = \sum_{\substack{s\equiv a\ (N)\\ t\equiv c\ (N)\\ s,t>0}} q^{\frac{st}{N}}$$

so that $F^*_{a,c,N} = K_{a,c,N} - K_{-a,-c,N}$. Suppose $\gcd(a, N) = d > 1$. If we take the sum over $c \equiv r \left(\bmod \frac{N}{d}\right)$ of the $F^*_{a,c,N}$ for some r, we observe

$$\sum_{\substack{0<c<N\\ c\equiv r\ \left(\frac{N}{d}\right)}} F^*_{a,c,N} = \sum_{\substack{0<c<N\\ c\equiv r\ \left(\frac{N}{d}\right)}} K_{a,c,N} - \sum_{\substack{0<c<N\\ c\equiv -r\ \left(\frac{N}{d}\right)}} K_{-a,c,N}$$

$$= K_{\frac{a}{d},r,\frac{N}{d}} - K_{-\frac{a}{d},-r,\frac{N}{d}}$$

$$= F^*_{\frac{a}{d},r,\frac{N}{d}}.$$

Of course, if $\frac{a}{d} \equiv -r \left(\bmod \frac{N}{d}\right)$ then this is 0.

5 The Identities

In this section we use what we have covered in the previous sections to rearrange the identities conjectured by Melham to find equivalent identities which can then be easily proven.

5.1 Triangular Numbers

5.1.1 Identity (6)

As mentioned before, Melham's identity 6 is

$$G_3(q)G_3(q^5) = \sum_{n=0}^{\infty}\left[\frac{q^{3n}+q^{7n+1}}{1-q^{20n+5}} - \frac{q^{13n+9}+q^{17n+12}}{1-q^{20n+15}}\right].$$

We expand the denominators in the RHS to give

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[q^{3n+(20n+5)m}+q^{7n+1+(20n+5)m}-q^{13n+9+(20n+15)m}-q^{17n+12+(20n+15)m}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[q^{\frac{(20n+5)(20m+3)}{20}-\frac{3}{4}}+q^{\frac{(20n+5)(20m+7)}{20}-\frac{3}{4}}-q^{\frac{(20n+15)(20m+13)}{20}-\frac{3}{4}}-q^{\frac{(20n+15)(20m+17)}{20}-\frac{3}{4}}\right]$$

$$=q^{-\frac{3}{4}}\left(\sum_{\substack{a,b>0\\ a\equiv 5\ (20)\\ b\equiv 3\ (20)}} q^{\frac{ab}{20}} - \sum_{\substack{a,b>0\\ a\equiv -5\ (20)\\ b\equiv -3\ (20)}} q^{\frac{ab}{20}} + \sum_{\substack{a,b>0\\ a\equiv 5\ (20)\\ b\equiv 7\ (20)}} q^{\frac{ab}{20}} - \sum_{\substack{a,b>0\\ a\equiv -5\ (20)\\ b\equiv -7\ (20)}} q^{\frac{ab}{20}}\right)$$

$$=q^{-\frac{3}{4}}\left(F^*_{5,3,20}+F^*_{5,7,20}\right).$$

While it may be desirable to include all of Melham's original identities here, the author would like to stress that some of them are very long, and adding them would increase the length of this work by around 10 pages. Instead, we expand the denominators in the RHS for all identities and include the form

$$G_k(q^\alpha)G_k(q^\beta) = q^{-\frac{\alpha+\beta}{8(k-2)}}\sum_{(a_i,c_i)} F^*_{a_i,c_i,4(k-2)},$$

for each of the identities instead, simplifying and shortening. We note by our earlier result that

$$\sum_{\substack{0<c<20\\ c\equiv 3\ (4)}} F^*_{5,c,20} = F^*_{1,3,4} = K_{1,3,4} - K_{3,1,4} = 0.$$

Hence

$$F^*_{5,3,20}+F^*_{5,7,20} = -F^*_{5,11,20} - F^*_{5,19,20}$$

as

$$F^*_{5,15,20} = 0.$$

Noting that $1, 4$ are quadratic residues modulo 5, where as $2, 3$ are not, we see the RHS is equal to, where $\left(\frac{c}{p}\right)$ denotes the Legendre symbol,

$$-\frac{q^{-\frac{3}{4}}}{2}\sum_{\substack{0<c<20\\ c\equiv 3\ (4)}}\left(\frac{c}{5}\right)F^*_{5,c,20},$$

equivalently,

$$\frac{q^{-\frac{3}{4}}}{2}\sum_{\substack{0<c<20\\ c\equiv 1\ (4)}}\left(\frac{c}{5}\right)F^*_{15,c,20}$$

$$=-\frac{q^{-\frac{3}{4}}}{80\pi i}\sum_{b=0}^{19}E_{15,b,20}\sum_{\substack{0<c<20\\ c\equiv 1\ (4)}}\left(\frac{c}{5}\right)\zeta_{20}^{-bc}.$$

Here we have used the fact that

$$F^*_{a,c,N}=-\frac{1}{2\pi iN}\sum_{b=0}^{N-1}\zeta_N^{-bc}E_{a,b,N}.$$

We will encounter sums of the form

$$\sum_{\substack{0<c<N\\ c\equiv\alpha\ (\beta)}}\left(\frac{c}{p}\right)\zeta_N^{-bc}$$

frequently. To deal with these we use a small lemma.

Lemma 5.1. *For $\alpha,\beta,p\in\mathbb{N}$, $\alpha<\beta$, and $N=\beta p$, $(\beta,p)=1$, p an odd prime, and let $\gamma\equiv\beta^{-1}$ modulo p. We have*

$$\sum_{\substack{0<c<N\\ c\equiv\alpha\ (\beta)}}\left(\frac{c}{p}\right)\zeta_N^{-bc}=\begin{cases}\zeta_N^{\alpha b(\gamma\beta-1)}\left(\frac{b\gamma}{p}\right)\sqrt{p} & p\equiv 1\ (mod\ 4)\\ -\zeta_N^{\alpha b(\gamma\beta-1)}\left(\frac{b\gamma}{p}\right)i\sqrt{p} & p\equiv 3\ (mod\ 4)\end{cases}$$

Proof. We have

$$S=\sum_{\substack{0<c<N\\ c\equiv\alpha\ (\beta)}}\left(\frac{c}{p}\right)\zeta_N^{-bc}=\sum_{k=0}^{\left\lfloor\frac{N-\alpha}{\beta}\right\rfloor}\left(\frac{\alpha+k\beta}{p}\right)\zeta_N^{-b(\alpha+k\beta)}=\zeta_N^{-b\alpha}\sum_{k=0}^{\left\lfloor\frac{N-\alpha}{\beta}\right\rfloor}\left(\frac{\alpha+k\beta}{p}\right)\zeta_p^{-bk}.$$

As $\alpha<\beta$ we have $\left\lfloor\frac{N-\alpha}{\beta}\right\rfloor=p-1$. Since $k\mapsto\alpha+k\beta$ is a bijection modulo p, letting $n=\alpha+k\beta$, we have $k\equiv(n-\alpha)\gamma\pmod p$, so

$$S=\zeta_N^{-\alpha b}\sum_{n=0}^{p-1}\left(\frac{n}{p}\right)\zeta_p^{-b(n-\alpha)\gamma}=\zeta_N^{-\alpha b}\zeta_p^{b\alpha\gamma}\sum_{n=0}^{p-1}\left(\frac{n}{p}\right)\zeta_N^{-b\gamma n}=\zeta_N^{\alpha b(\gamma\beta-1)}\sum_{n=0}^{p-1}\left(\frac{n}{p}\right)\zeta_N^{-b\gamma n}.$$

This is a Gauss sum. Gauss proved [7] that

$$\sum_{n=0}^{p-1}\left(\frac{n}{p}\right)\zeta_p^{-bn}=\begin{cases}\left(\frac{b}{p}\right)\sqrt{p} & p\equiv 1\pmod 4\\ -\left(\frac{b}{p}\right)i\sqrt{p} & p\equiv 3\pmod 4\end{cases}$$

for p an odd prime. We can therefore conclude with the statement of the lemma. □

Now, returning to the RHS of identity (6), we have

$$\sum_{\substack{0<c<20\\ c\equiv 1\ (4)}}\left(\frac{c}{5}\right)\zeta_{20}^{-bc}=\left(\frac{b}{5}\right)\sqrt{5}.$$

Hence we see the RHS is

$$-\frac{\sqrt{5}}{80\pi i} q^{-\frac{3}{4}} \sum_{b=0}^{19} \left(\frac{b}{5}\right) \zeta_{20}^{15b} E_{15,b,20}(\tau).$$

From before, we have

$$\frac{1}{4} q^{-\frac{\alpha+\beta}{8}} \vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right) = G_3(q^\alpha) G_3(q^\beta)$$

with $\Gamma = \left\langle \begin{pmatrix} \sqrt{2\alpha} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2\beta} \end{pmatrix} \right\rangle$, with $\rho = \begin{pmatrix} \sqrt{\frac{\alpha}{2}} \\ \sqrt{\frac{\beta}{2}} \end{pmatrix} \in \Gamma^*$. Hence identity (6) is equivalent to

$$\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right) = -\frac{\sqrt{5}}{20\pi i} \sum_{b=0}^{19} \left(\frac{b}{5}\right) \zeta_{20}^{15b} E_{15,b,20}(\tau).$$

Raising both sides to the power of 2, as $\alpha + \beta = 1 + 5 = 6$ is even, we see the LHS is a modular form for the congruence subgroup $\Gamma_0(10) \cap \Gamma^0(2)$, as detailed before in section 3. Thus we aim to show that the RHS (ignoring the constant term),

$$H(\tau) := \left(\sum_{b=0}^{19} \left(\frac{b}{5}\right) \zeta_{20}^{15b} E_{15,b,20} \right)^2$$

is also a modular form for this subgroup. We will see to this matter in the next section, but first we have the other identities to transform into a similar form.

5.1.2 Identity (7)

Melham states the following:

$$G_3(q) G_3(q^6) = \sum_{n=0}^{\infty} \left[\frac{q^{7n}}{1-q^{24n+3}} + \frac{q^{5n+1}}{1-q^{24n+9}} - \frac{q^{19n+11}}{1-q^{24n+15}} - \frac{q^{17n+14}}{1-q^{24n+21}} \right].$$

We expand the denominators in the RHS to give

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[q^{7n+(24n+3)m} + q^{5n+1+(24n+9)m} - q^{19n+11+(24n+15)m} - q^{17n+14+(24n+21)m} \right]$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[q^{\frac{(24n+3)(24m+7)}{24} - \frac{7}{8}} + q^{\frac{(24n+9)(24m+5)}{24} - \frac{7}{8}} - q^{\frac{(24n+15)(24m+19)}{24} - \frac{7}{8}} - q^{\frac{(24n+21)(24m+17)}{24} - \frac{7}{8}} \right]$$

$$= q^{-\frac{7}{8}} \left(\sum_{\substack{a,b>0 \\ a\equiv 3\ (24) \\ b\equiv 7\ (24)}} q^{\frac{ab}{24}} - \sum_{\substack{a,b>0 \\ a\equiv -3\ (24) \\ b\equiv -7\ (24)}} q^{\frac{ab}{24}} + \sum_{\substack{a,b>0 \\ a\equiv 9\ (24) \\ b\equiv 5\ (24)}} q^{\frac{ab}{24}} - \sum_{\substack{a,b>0 \\ a\equiv -9\ (24) \\ b\equiv -5\ (24)}} q^{\frac{ab}{24}} \right)$$

$$= q^{-\frac{7}{8}} \left(F^*_{3,7,24} + F^*_{9,5,24} \right).$$

As

$$\sum_{\substack{0<c<24 \\ c\equiv 7\ (8)}} F^*_{3,c,24} = 3F^*_{1,7,8} = 0$$

and

$$\sum_{\substack{0<c<24 \\ c\equiv 5\ (8)}} F^*_{9,c,24} = 3F^*_{3,5,8} = 0,$$

we obtain

$$F^*_{3,7,24} + F^*_{9,5,24} = -F^*_{3,15,24} - F^*_{3,23,24} - F^*_{9,13,24} - F^*_{9,21,24}.$$

Noticing that $F^*_{3,15,24} + F^*_{9,21,24} = 0$, we have

$$F^*_{3,7,24} + F^*_{9,5,24} = -F^*_{3,23,24} - F^*_{9,13,24}$$

hence the RHS is equal to

$$\frac{q^{-\frac{7}{8}}}{2}\left(\sum_{\substack{0<c<24\\ c\equiv 7\ (8)}}\left(\frac{c}{3}\right)F^*_{3,c,24} - \sum_{\substack{0<c<24\\ c\equiv 5\ (8)}}\left(\frac{c}{3}\right)F^*_{9,c,24}\right).$$

In the pursuit of consistency, we would like the second sum to run over $c \equiv 3 \pmod 8$. We have

$$-\sum_{\substack{0<c<24\\ c\equiv 5\ (8)}}\left(\frac{c}{3}\right)F^*_{9,c,24} = \sum_{\substack{0<c<24\\ c\equiv 5\ (8)}}\left(\frac{c}{3}\right)F^*_{15,-c,24} = \sum_{\substack{0<c<24\\ -c\equiv 5\ (8)}}\left(\frac{-c}{3}\right)F^*_{15,c,24} = -\sum_{\substack{0<c<24\\ c\equiv 3\ (8)}}\left(\frac{c}{3}\right)F^*_{15,c,24}$$

as $\left(\frac{-1}{3}\right) = -1$. Therefore the RHS becomes

$$\frac{q^{-\frac{7}{8}}}{2}\left(\sum_{\substack{0<c<24\\ c\equiv 7\ (8)}}\left(\frac{c}{3}\right)F^*_{3,c,24} - \sum_{\substack{0<c<24\\ c\equiv 3\ (8)}}\left(\frac{c}{3}\right)F^*_{15,c,24}\right)$$

$$= \frac{\sqrt{3}}{96\pi}q^{-\frac{7}{8}}\sum_{b=0}^{23}\left(\frac{b}{3}\right)\left(\zeta_{24}^{9b}E_{3,b,24} - \zeta_{24}^{21b}E_{15,b,24}\right)$$

by our previous lemma. Thus identity 7 is equivalent to

$$\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right) = \frac{\sqrt{3}}{24\pi}\sum_{b=0}^{23}\left(\frac{b}{3}\right)\left(\zeta_{24}^{9b}E_{3,b,24} - \zeta_{24}^{21b}E_{15,b,24}\right)$$

and we know, as $\alpha + \beta = 1 + 6 = 7$ is odd, that

$$\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right)^4$$

is a modular form for $\Gamma_0(12) \cap \Gamma^0(2)$. Thus we would like to show that the RHS,

$$H(\tau) = \left(\sum_{b=0}^{23}\left(\frac{b}{3}\right)\left(\zeta_{24}^{9b}E_{3,b,24} - \zeta_{24}^{21b}E_{15,b,24}\right)\right)^4$$

is also a modular form for this subgroup. As all the following identities transform in a similar way, the details have mostly been suppressed from here on.

5.1.3 Identity (8)

Melham states the following:

$$G_3(q^2)G_3(q^3) = \sum_{n=0}^{\infty}\left[\frac{q^{5n}}{1-q^{24n+3}} + \frac{q^{7n+2}}{1-q^{24n+9}} - \frac{q^{17n+10}}{1-q^{24n+15}} - \frac{q^{19n+16}}{1-q^{24n+21}}\right].$$

As before, we find the RHS is equal to

$$q^{-\frac{5}{8}}\left(F^*_{3,5,24}+F^*_{9,7,24}\right)$$

and therefore the identity is equivalent to

$$\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right)=\frac{\sqrt{3}}{24\pi}\sum_{b=0}^{23}\left(\frac{b}{3}\right)\left(\zeta_{24}^{21b}E_{9,b,24}-\zeta_{24}^{9b}E_{21,b,24}\right).$$

Hence we need to show that

$$H(\tau):=\left(\sum_{b=0}^{23}\left(\frac{b}{3}\right)\left(\zeta_{24}^{21b}E_{9,b,24}-\zeta_{24}^{9b}E_{21,b,24}\right)\right)^4$$

is modular for $\Gamma_0(12)\cap\Gamma^0(2)$.

5.1.4 Identity (9)

Melham states the following:

$$G_3(q)G_3(q^{10})=\sum_{n=0}^{\infty}\left[\frac{q^{11n}+q^{19n+1}}{1-q^{40n+5}}-\frac{q^{17n+5}+q^{33n+11}}{1-q^{40n+15}}+\frac{q^{7n+3}+q^{23n+13}}{1-q^{40n+25}}-\frac{q^{21n+17}+q^{29n+24}}{1-q^{40n+35}}\right].$$

By our earlier method, the RHS is

$$q^{-\frac{11}{8}}\left(F^*_{5,11,40}+F^*_{5,19,40}+F^*_{25,7,40}+F^*_{25,23,40}\right)$$

and so the identity becomes

$$\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right)=\frac{\sqrt{5}}{40\pi i}\sum_{b=0}^{39}\left(\frac{b}{5}\right)\left(\zeta_{40}^{5b}E_{5,b,40}-\zeta_{40}^{25b}E_{25,b,40}\right).$$

Hence we need to show that

$$H(\tau):=\left(\sum_{b=0}^{39}\left(\frac{b}{5}\right)\left(\zeta_{40}^{5b}E_{5,b,40}-\zeta_{40}^{25b}E_{25,b,40}\right)\right)^4$$

is modular for $\Gamma_0(20)\cap\Gamma^0(2)$.

5.1.5 Identity (10)

Using our earlier method, the identity is equivalent to

$$G_3(q^2)G_3(q^5)=q^{-\frac{7}{8}}\left(F^*_{5,7,40}+F^*_{5,23,40}+F^*_{25,11,40}+F^*_{25,19,40}\right)$$

and

$$\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right)=\frac{\sqrt{5}}{40\pi i}\sum_{b=0}^{39}\left(\frac{b}{5}\right)\left(\zeta_{40}^{5b}E_{25,b,40}-\zeta_{40}^{25b}E_{5,b,40}\right).$$

So

$$H(\tau):=\left(\sum_{b=0}^{39}\left(\frac{b}{5}\right)\left(\zeta_{40}^{5b}E_{25,b,40}-\zeta_{40}^{25b}E_{5,b,40}\right)\right)^4.$$

5.1.6 Identity (11)

Again we use our earlier method and find the identity is equivalent to

$$G_3(q)G_3(q^{13}) = q^{-\frac{7}{4}}\left(F^*_{13,7,52} + F^*_{13,11,52} + F^*_{13,15,52} + F^*_{13,19,52} + F^*_{13,31,52} + F^*_{13,47,52}\right).$$

So

$$H(\tau) := \left(\sum_{b=0}^{51}\left(\frac{b}{13}\right)\zeta_{52}^{39b}E_{39,b,52}\right)^2.$$

5.1.7 Identity (12)

Again we use our earlier method and find the identity is equivalent to

$$\begin{aligned}G_3(q)G_3(q^{22}) = q^{-\frac{23}{8}}\big(&F^*_{11,15,88} + F^*_{11,23,88} + F^*_{11,31,88} + F^*_{11,47,88} + F^*_{11,71,88}\\ &+F^*_{55,11,88} + F^*_{55,19,88} + F^*_{55,35,88} + F^*_{55,43,88} + F^*_{55,51,88} + F^*_{55,83,88}\big).\end{aligned}$$

So

$$H(\tau) := \left(\sum_{b=0}^{87}\left(\frac{b}{11}\right)\left(\zeta_{88}^{33b}E_{11,b,88} - \zeta_{88}^{77b}E_{55,b,88}\right)\right)^4.$$

5.1.8 Identity (13)

This identity is slightly resistant to our previous method, and requires a little more effort. Recalling that $K_{a,c,N} - K_{-a,-c,N} = -\frac{1}{2\pi iN}F_{a,c,N}$, we also define $F^*_{a,c,N} := -\frac{1}{2\pi iN}F_{a,c,N}$. As before, we expand the denominators, and find the identity is equivalent to

$$\begin{aligned}G_3(q^2)G_3(q^{11}) = q^{-\frac{13}{8}}\Big(&-K_{11,5,88} - K_{11,37,88} - K_{11,45,88} - K_{11,53,88} - K_{11,69,88}\\ &-K_{77,3,88} - K_{77,11,88} - K_{77,27,88} - K_{77,59,88} - K_{77,67,88} - K_{77,75,88}\\ &+K_{33,15,88} + K_{33,23,88} + K_{33,31,88} + K_{33,47,88} + K_{33,71,88}\\ &+K_{55,1,88} + K_{55,9,88} + K_{55,25,88} + K_{55,33,88} + K_{55,49,88} + K_{55,81,88}\Big).\end{aligned}$$

Now, noting that we can write $-K_{-a,-c,N} = F^*_{a,c,N} - K_{a,c,N}$, and also $K_{a,c,N} = F^*_{a,c,N} + K_{-a,-c,N}$ we have this being equivalent to, multiplying by $q^{\frac{13}{8}}$ for simplicity,

$$\begin{aligned}q^{\frac{13}{8}}G_3(q^2)G_3(q^{11}) =& -K_{11,5,88} - K_{11,37,88} - K_{11,45,88} - K_{11,53,88} - K_{11,69,88}\\ &+(F^*_{11,85,88} - K_{11,85,88}) + (F^*_{11,77,88} - K_{11,77,88}) + (F^*_{11,61,88} - K_{11,61,88}) + (F^*_{11,29,88} - K_{11,29,88})\\ &+(F^*_{11,21,88} - K_{11,21,88}) + (F^*_{11,13,88} - K_{11,13,88})\\ &+(F^*_{33,15,88} + K_{55,73,88}) + (F^*_{33,23,88} + K_{55,65,88}) + (F^*_{33,31,88} + K_{55,57,88}) + (F^*_{33,47,88} + K_{55,41,88})\\ &+(F^*_{33,71,88} + K_{55,17,88})\\ &+K_{55,1,88} + K_{55,9,88} + K_{55,25,88} + K_{55,33,88} + K_{55,49,88} + K_{55,81,88}\end{aligned}$$

$$
\begin{aligned}
&= F^*_{11,85,88} + F^*_{11,77,88} + F^*_{11,61,88} + F^*_{11,29,88} + F^*_{11,21,88} + F^*_{11,13,88} \\
&\quad + F^*_{33,15,88} + F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88} \\
&\quad - K_{11,5,88} - K_{11,13,88} - K_{11,21,88} - K_{11,29,88} - K_{11,37,88} - K_{11,45,88} - K_{11,53,88} - K_{11,61,88} \\
&\quad - K_{11,69,88} - K_{11,77,88} - K_{11,85,88} \\
&\quad + K_{55,1,88} + K_{55,9,88} + K_{55,17,88} + K_{55,25,88} + K_{55,33,88} + K_{55,41,88} + K_{55,49,88} + K_{55,57,88} \\
&\quad + K_{55,65,88} + K_{55,73,88} + K_{55,81,88} \\
&= F^*_{11,85,88} + F^*_{11,77,88} + F^*_{11,61,88} + F^*_{11,29,88} + F^*_{11,21,88} + F^*_{11,13,88} + F^*_{33,15,88} \\
&\quad + F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88} - \sum_{r=0}^{10} K_{11,5+8r,88} + \sum_{r=0}^{10} K_{55,1+8r,88} \\
&= F^*_{11,85,88} + F^*_{11,77,88} + F^*_{11,61,88} + F^*_{11,29,88} + F^*_{11,21,88} + F^*_{11,13,88} + F^*_{33,15,88} \\
&\quad + F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88} - K_{1,5,8} + K_{5,1,8} \\
&= F^*_{11,85,88} + F^*_{11,77,88} + F^*_{11,61,88} + F^*_{11,29,88} + F^*_{11,21,88} + F^*_{11,13,88} + F^*_{33,15,88} \\
&\quad + F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88}.
\end{aligned}
$$

Hence identity (13) is equivalent to

$$
\begin{aligned}
G_3(q^2)G_3(q^{11}) &= q^{-\frac{13}{8}}\Big(F^*_{11,13,88} + F^*_{11,21,88} + F^*_{11,29,88} + F^*_{11,61,88} + F^*_{11,85,88} \\
&\quad + F^*_{33,15,88} + F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88}\Big) \\
&= \frac{1}{2}q^{-\frac{13}{8}}\left(\sum_{\substack{0<c<88\\ c\equiv 7\ (8)}}\left(\frac{c}{11}\right)F^*_{33,c,88} - \sum_{\substack{0<c<88\\ c\equiv 3\ (8)}}\left(\frac{c}{11}\right)F^*_{77,c,88}\right)
\end{aligned}
$$

as -1 is not a quadratic residue modulo 11. This is therefore the same as

$$
G_3(q^2)G_3(q^{11}) = \frac{\sqrt{11}}{352\pi}q^{-\frac{13}{8}}\sum_{b=0}^{87}\left(\frac{b}{11}\right)\left(\zeta_{88}^{33b}E_{33,b,88} - \zeta_{88}^{77b}E_{77,b,88}\right)
$$

which becomes

$$
\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right) = \frac{\sqrt{11}}{88\pi}\sum_{b=0}^{87}\left(\frac{b}{11}\right)\left(\zeta_{88}^{33b}E_{33,b,88} - \zeta_{88}^{77b}E_{77,b,88}\right).
$$

Hence

$$
H(\tau) := \left(\sum_{b=0}^{87}\left(\frac{b}{11}\right)\left(\zeta_{88}^{33b}E_{33,b,88} - \zeta_{88}^{77b}E_{77,b,88}\right)\right)^4.
$$

5.1.9 Identity (14)

We find, using our earlier method, that the identity is equivalent to

$$
\begin{aligned}
G_3(q)G_3(q^{37}) &= q^{-\frac{19}{4}}\Big(F^*_{37,15,148} + F^*_{37,19,148} + F^*_{37,23,148} + F^*_{37,31,148} + F^*_{37,35,148} \\
&\quad + F^*_{37,39,148} + F^*_{37,43,148} + F^*_{37,51,148} + F^*_{37,55,148} + F^*_{37,59,148} + F^*_{37,79,148} + F^*_{37,87,148} \\
&\quad + F^*_{37,91,148} + F^*_{37,103,148} + F^*_{37,119,148} + F^*_{37,131,148} + F^*_{37,135,148} + F^*_{37,143,148}\Big)
\end{aligned}
$$

so

$$
H(\tau) := \left(\sum_{b=0}^{147}\left(\frac{b}{37}\right)\zeta_{148}^{111b}E_{111,b,148}\right)^2.
$$

5.1.10 Identity (15)

Using the same approach of Identity (13), we find the identity to be equivalent to

$$\begin{aligned}G_3(q)G_3(q^{58}) = q^{-\frac{59}{8}}\Big(&F^*_{29,35,232}+F^*_{29,51,232}+F^*_{29,59,232}+F^*_{29,67,232}+F^*_{29,83,232}+F^*_{29,91,232}\\&+F^*_{29,107,232}+F^*_{29,115,232}+F^*_{29,123,232}+F^*_{29,139,232}+F^*_{29,179,232}+F^*_{29,187,232}\\&+F^*_{29,203,232}+F^*_{29,219,232}+F^*_{29,227,232}\\&+F^*_{145,15,232}+F^*_{145,31,232}+F^*_{145,39,232}+F^*_{145,47,232}+F^*_{145,55,232}+F^*_{145,79,232}+F^*_{145,87,232}+F^*_{145,95,232}\\&+F^*_{145,119,232}+F^*_{145,127,232}+F^*_{145,135,232}+F^*_{145,143,232}+F^*_{145,159,232}+F^*_{145,191,232}+F^*_{145,215,232}\Big)\end{aligned} \tag{1}$$

$$= \frac{1}{2}q^{-\frac{59}{8}}\left(\sum_{\substack{0<c<232\\ c\equiv 3\ (8)}}\left(\frac{c}{29}\right)F^*_{29,c,232} - \sum_{\substack{0<c<232\\ c\equiv 7\ (8)}}\left(\frac{c}{29}\right)F^*_{145,c,232}\right) \tag{2}$$

$$= \frac{\sqrt{29}}{928\pi i}q^{-\frac{59}{8}}\sum_{b=0}^{231}\left(\frac{b}{29}\right)\left(\zeta_{232}^{29b}E_{29,b,232}-\zeta_{232}^{145b}E_{145,b,232}\right).$$

Hence

$$H(\tau) := \left(\sum_{b=0}^{231}\left(\frac{b}{29}\right)\left(\zeta_{232}^{29b}E_{29,b,232}-\zeta_{232}^{145b}E_{145,b,232}\right)\right)^4$$

5.1.11 Identity (16)

Using our earlier approach, we find identity (16) to be equivalent to

$$\begin{aligned}G_3(q^2)G_3(q^{29}) = q^{-\frac{31}{8}}\Big(&F^*_{29,15,232}+F^*_{29,31,232}+F^*_{29,39,232}+F^*_{29,47,232}+F^*_{29,55,232}+F^*_{29,79,232}\\&+F^*_{29,95,232}+F^*_{29,119,232}+F^*_{29,127,232}+F^*_{29,135,232}+F^*_{29,143,232}+F^*_{29,159,232}+F^*_{29,191,232}+F^*_{29,215,232}\\&+F^*_{145,35,232}+F^*_{145,51,232}+F^*_{145,59,232}+F^*_{145,67,232}+F^*_{145,83,232}+F^*_{145,91,232}+F^*_{145,107,232}\\&+F^*_{145,115,232}+F^*_{145,123,232}+F^*_{145,139,232}+F^*_{145,179,232}+F^*_{145,187,232}+F^*_{145,219,232}+F^*_{145,227,232}\Big)\end{aligned}$$

and

$$H(\tau) := \left(\sum_{b=0}^{231}\left(\frac{b}{29}\right)\left(\zeta_{232}^{145b}E_{29,b,232}-\zeta_{232}^{29b}E_{145,b,232}\right)\right)^4.$$

5.2 Pentagonal Numbers

5.2.1 Identity (17)

Again we use our earlier approach, and find the identity is equivalent to

$$\begin{aligned}G_5(q)G_5(q^5) = q^{-\frac{1}{4}}\big(&F^*_{15,1,60}+F^*_{15,29,60}+F^*_{15,41,60}+F^*_{15,49,60}\\&+F^*_{45,7,60}+F^*_{45,23,60}+F^*_{45,43,60}+F^*_{45,47,60}\big).\end{aligned}$$

This identity is a little unusual, but we notice we can write it as

$$\begin{aligned}G_5(q)G_5(q^5) = q^{-\frac{1}{4}}\big[&F^*_{15,1,60}+F^*_{15,29,60}+F^*_{15,41,60}+F^*_{15,49,60}\\ &-\big(F^*_{15,53,60}+F^*_{15,37,60}+F^*_{15,17,60}+F^*_{15,13,60}\big)\big]\\ =\ & q^{-\frac{1}{4}}\left(\sum_{\substack{0<c<60\\ c\equiv 1\ (12)}}\left(\frac{c}{5}\right)F^*_{15,c,60}+\sum_{\substack{0<c<60\\ c\equiv 5\ (12)}}\left(\frac{c}{5}\right)F^*_{15,c,60}\right)\\ =\ & q^{-\frac{1}{4}}\left(\sum_{b=0}^{59}E_{15,b,60}\sum_{\substack{0<c<60\\ c\equiv 1\ (12)}}\left(\frac{c}{5}\right)\zeta_{60}^{-bc}+\sum_{b=0}^{59}E_{15,b,60}\sum_{\substack{0<c<60\\ c\equiv 5\ (12)}}\left(\frac{c}{5}\right)\zeta_{60}^{-bc}\right)\\ =\ & q^{-\frac{1}{4}}\sum_{b=0}^{59}E_{15,b,60}\left(\sum_{\substack{0<c<60\\ c\equiv 1\ (12)}}\left(\frac{c}{5}\right)\zeta_{60}^{-bc}+\sum_{\substack{0<c<60\\ c\equiv 5\ (12)}}\left(\frac{c}{5}\right)\zeta_{60}^{-bc}\right)\\ =\ & \frac{\sqrt{5}}{120\pi i}q^{-\frac{1}{4}}\sum_{b=0}^{59}\left[\left(\frac{b}{5}\right)\left(\zeta_{60}^{35b}+\zeta_{60}^{55b}\right)E_{15,b,60}\right].\end{aligned}$$

Hence

$$H(\tau) := \left(\sum_{b=0}^{59}\left(\frac{b}{5}\right)\left(\zeta_{60}^{35b}E_{15,b,60}-\zeta_{60}^{5b}E_{45,b,60}\right)\right)^2.$$

5.2.2 Identity (18)

Using the approach of identity (13), we find the identity is equivalent to

$$\begin{aligned}G_5(q)G_5(q^{10}) = q^{-\frac{11}{24}}\big(&F^*_{5,11,120}+F^*_{5,35,120}+F^*_{5,59,120}\\ &+F^*_{35,53,120}+F^*_{35,77,120}\\ &+F^*_{65,23,120}+F^*_{65,47,120}\\ &+F^*_{95,41,120}+F^*_{95,65,120}+F^*_{95,89,120}\big).\end{aligned}$$

Therefore

$$H(\tau) := \left(\sum_{b=0}^{119}\left(\frac{b}{5}\right)\left(\zeta_{120}^{105b}E_{25,b,120}-\zeta_{120}^{45b}E_{85,b,120}+\zeta_{120}^{105b}E_{65,b,120}-\zeta_{120}^{45b}E_{5,b,120}\right)\right)^{12},$$

recalling that our exponent comes from the requirement in section 3 for the LHS to be a modular form on the desired subgroup.

5.2.3 Identity (19)

Using the approach of identity (13), we find the identity is equivalent to

$$\begin{aligned}G_5(q^2)G_5(q^5) = q^{-\frac{7}{24}}\big(&F^*_{5,7,120}+F^*_{5,103,120}\\ &+F^*_{25,11,120}+F^*_{25,59,120}\\ &+F^*_{65,19,120}+F^*_{65,91,120}\\ &+F^*_{85,23,120}+F^*_{85,47,120}\big).\end{aligned}$$

So we define

$$H(\tau) := \left(\sum_{b=0}^{119} \left(\frac{b}{5}\right)\left(\zeta_{120}^{105b}E_{5,b,120} - \zeta_{120}^{45b}E_{65,b,120} + \zeta_{120}^{105b}E_{85,b,120} - \zeta_{120}^{45b}E_{25,b,120}\right)\right)^{12}.$$

5.2.4 Identity (20)

Here we use our earlier approach, and see the identity is equivalent to

$$\begin{aligned}G_5(q)G_5(q^{13}) = q^{-\frac{7}{12}}\big(&F^*_{13,7,156} + F^*_{13,19,156} + F^*_{13,31,156} + F^*_{13,67,156} + F^*_{13,115,156} + F^*_{13,151,156}\\ &+F^*_{65,11,156} + F^*_{65,47,156} + F^*_{65,59,156} + F^*_{65,71,156} + F^*_{65,83,156} + F^*_{65,119,156}\big).\end{aligned}$$

So we let

$$H(\tau) := \left(\sum_{b=0}^{155} \left(\frac{b}{13}\right)\left(\zeta_{156}^{143b}E_{91,b,156} - \zeta_{156}^{65b}E_{13,b,156}\right)\right)^{6}.$$

5.2.5 Identity (21)

Our earlier approach shows us that the identity is equivalent to

$$\begin{aligned}G_5(q)G_5(q^{22}) = q^{-\frac{23}{24}}\big(&F^*_{143,35,264} + F^*_{143,83,264} + F^*_{143,107,264} + F^*_{143,131,264} + F^*_{143,227,264}\\ &+F^*_{187,31,264} + F^*_{187,103,264} + F^*_{187,199,264} + F^*_{187,223,264} + F^*_{187,247,264}\\ &+F^*_{209,29,264} + F^*_{209,101,264} + F^*_{209,149,264} + F^*_{209,173,264} + F^*_{209,197,264}\\ &+F^*_{253,1,264} + F^*_{253,25,264} + F^*_{253,49,264} + F^*_{253,97,264} + F^*_{253,169,264}\big).\end{aligned}$$

Thus

$$H(\tau) := \left(\sum_{b=0}^{263} \left(\frac{b}{11}\right)\left(\zeta_{264}^{121b}E_{11,b,264} - \zeta_{264}^{253b}E_{143,b,264} + \zeta_{264}^{209b}E_{187,b,264} - \zeta_{264}^{187b}E_{209,b,264}\right)\right)^{12}.$$

5.2.6 Identity (22)

Our earlier approach shows us that the identity is equivalent to

$$\begin{aligned}G_5(q^2)G_5(q^{11}) = q^{-\frac{13}{24}}\big(&F^*_{11,13,264} + F^*_{11,61,264} + F^*_{11,85,264} + F^*_{11,109,264} + F^*_{11,205,264}\\ &+F^*_{121,23,264} + F^*_{121,47,264} + F^*_{121,71,264} + F^*_{121,119,264} + F^*_{121,191,264}\\ &+F^*_{187,29,264} + F^*_{187,101,264} + F^*_{187,149,264} + F^*_{187,173,264} + F^*_{187,197,264}\\ &+F^*_{209,31,264} + F^*_{209,103,264} + F^*_{209,199,264} + F^*_{209,223,264} + F^*_{209,247,264}\big).\end{aligned}$$

We therefore define

$$H(\tau) := \left(\sum_{b=0}^{263} \left(\frac{b}{11}\right)\left(\zeta_{264}^{121b}E_{121,b,264} - \zeta_{264}^{187b}E_{187,b,264} + \zeta_{264}^{209b}E_{209,b,264} - \zeta_{264}^{253b}E_{253,b,264}\right)\right)^{12}.$$

5.2.7 Identity (23)

For this identity, we must use the approach of identity (13), and find equivalence with

$$\begin{aligned}G_5(q)G_5(q^{37}) = q^{-\frac{19}{12}}\big(&F^*_{37,19,444} + F^*_{37,31,444} + F^*_{37,43,444} + F^*_{37,55,444} + F^*_{37,79,444} + F^*_{37,91,444}\\
&+F^*_{37,103,444} + F^*_{37,163,444} + F^*_{37,187,444} + F^*_{37,199,444} + F^*_{37,235,444} + F^*_{37,283,444}\\
&+F^*_{37,319,444} + F^*_{37,331,444} + F^*_{37,355,444} + F^*_{37,415,444} + F^*_{37,427,444} + F^*_{37,439,444}\\
&+F^*_{185,23,444} + F^*_{185,35,444} + F^*_{185,59,444}\\
&+F^*_{185,119,444} + F^*_{185,131,444} + F^*_{185,143,444} + F^*_{185,167,444} + F^*_{185,179,444} + F^*_{185,191,444}\\
&+F^*_{185,203,444} + F^*_{185,227,444} + F^*_{185,239,444} + F^*_{185,251,444}\\
&+F^*_{185,311,444} + F^*_{185,335,444} + F^*_{185,347,444} + F^*_{185,383,444} + F^*_{185,431,444}\big).\end{aligned}$$

We therefore let

$$H(\tau) := \left(\sum_{b=0}^{443}\left(\frac{b}{37}\right)\left(\zeta_{444}^{407b}E_{259,b,444} - \zeta_{444}^{185b}E_{37,b,444}\right)\right)^6.$$

5.2.8 Identity (24)

Again, we use the approach of identity (13), and find identity (24) is equivalent to

$$\begin{aligned}G_5(q)G_5(q^{58}) = q^{-\frac{59}{24}}\big(&F^*_{29,35,696} + F^*_{29,59,696} + F^*_{29,83,696} + F^*_{29,107,696} + F^*_{29,179,696}\\
&+F^*_{29,203,696} + F^*_{29,227,696} + F^*_{29,299,696} + F^*_{29,323,696} + F^*_{29,347,696} + F^*_{29,371,696}\\
&+F^*_{29,419,696} + F^*_{29,515,696} + F^*_{29,587,696} + F^*_{29,683,696}\\
&+F^*_{203,77,696} + F^*_{203,101,696} + F^*_{203,221,696} + F^*_{203,269,696} + F^*_{203,293,696}\\
&+F^*_{203,317,696} + F^*_{203,365,696} + F^*_{203,389,696} + F^*_{203,437,696} + F^*_{203,461,696} + F^*_{203,485,696}\\
&+F^*_{203,533,696} + F^*_{203,653,696} + F^*_{203,677,696}\\
&+F^*_{377,47,696} + F^*_{377,95,696} + F^*_{377,119,696} + F^*_{377,143,696} + F^*_{377,191,696}\\
&+F^*_{377,215,696} + F^*_{377,263,696} + F^*_{377,287,696} + F^*_{377,311,696} + F^*_{377,359,696}\\
&+F^*_{377,479,696} + F^*_{377,503,696} + F^*_{377,599,696} + F^*_{377,623,696}\\
&+F^*_{551,65,696} + F^*_{551,161,696} + F^*_{551,209,696} + F^*_{551,233,696} + F^*_{551,257,696} + F^*_{551,281,696}\\
&+F^*_{551,353,696} + F^*_{551,377,696} + F^*_{551,401,696} + F^*_{551,473,696} + F^*_{551,497,696}\\
&+F^*_{551,521,696} + F^*_{551,545,696} + F^*_{551,593,696} + F^*_{551,689,696}\big).\end{aligned}$$

So we define

$$H(\tau) := \left(\sum_{b=0}^{695}\left(\frac{b}{29}\right)\left(\zeta_{696}^{493b}E_{29,b,696} - \zeta_{696}^{145b}E_{377,b,696} + \zeta_{696}^{319b}E_{551,b,696} - \zeta_{696}^{667b}E_{203,b,696}\right)\right)^{12}.$$

5.2.9 Identity (25)

Using the approach of identity (13), we see the identity is equivalent with

$$\begin{aligned}G_5(q^2)G_5(q^{29}) = q^{-\frac{31}{24}}\big(&F^*_{29,31,696} + F^*_{29,55,696} + F^*_{29,79,696} + F^*_{29,127,696}\\
&+F^*_{29,247,696} + F^*_{29,271,696} + F^*_{29,319,696} + F^*_{29,367,696} + F^*_{29,391,696}\\
&+F^*_{29,511,696} + F^*_{29,559,696} + F^*_{29,583,696} + F^*_{29,607,696} + F^*_{29,655,696} + F^*_{29,679,696}\\
&+F^*_{319,77,696} + F^*_{319,101,696} + F^*_{319,221,696} + F^*_{319,269,696} + F^*_{319,293,696}\\
&+F^*_{319,317,696} + F^*_{319,365,696} + F^*_{319,389,696}\\
&+F^*_{319,437,696} + F^*_{319,461,696} + F^*_{319,485,696} + F^*_{319,533,696} + F^*_{319,653,696} + F^*_{319,677,696}\\
&+F^*_{493,47,696} + F^*_{493,95,696} + F^*_{493,119,696} + F^*_{493,143,696} + F^*_{493,191,696}\\
&+F^*_{493,215,696} + F^*_{493,263,696} + F^*_{493,287,696} + F^*_{493,311,696} + F^*_{493,359,696}\\
&+F^*_{493,479,696} + F^*_{493,503,696} + F^*_{493,551,696} + F^*_{493,599,696} + F^*_{493,623,696}\\
&+F^*_{551,37,696} + F^*_{551,61,696} + F^*_{551,85,696} + F^*_{551,133,696} + F^*_{551,157,696}\\
&+F^*_{551,205,696} + F^*_{551,229,696} + F^*_{551,253,696} + F^*_{551,301,696}\\
&+F^*_{551,421,696} + F^*_{551,445,696} + F^*_{551,541,696} + F^*_{551,565,696} + F^*_{551,685,696}\big).\end{aligned}$$

So we let

$$H(\tau) := \left(\sum_{b=0}^{695}\left(\frac{b}{29}\right)\left(\zeta_{696}^{319b}E_{667,b,696} - \zeta_{696}^{667b}E_{319,b,696} + \zeta_{696}^{493b}E_{145,b,696} - \zeta_{696}^{145b}E_{493,b,696}\right)\right)^{12}.$$

5.3 Heptagonal Numbers

5.3.1 Identity (26)

Our final identity, and the only one involving the heptagonal numbers, requires us to use the approach of identity (13). It is our most complex identity in terms of length. We find identity (26) to be equivalent to

$$\begin{aligned}G_7(q)G_7(q^6) &= q^{-\frac{63}{40}}\Big(F^*_{3,103,120} + F^*_{9,21,120} + F^*_{9,101,120}\\
&\quad+F^*_{21,49,120} + F^*_{33,53,120} + F^*_{33,93,120}\\
&\quad+F^*_{51,79,120} + F^*_{57,77,120} + F^*_{81,29,120} + F^*_{93,73,120}\Big)\\
&= -\frac{1}{2}q^{-\frac{63}{40}}\Bigg(\sum_{\substack{0<c<120\\ c\equiv 23\ (40)}}\left(\frac{c}{3}\right)F^*_{3,c,120} - \sum_{\substack{0<c<120\\ c\equiv 21\ (40)}}\left(\frac{c}{3}\right)F^*_{9,c,120} + \sum_{\substack{0<c<120\\ c\equiv 9\ (40)}}\left(\frac{c}{3}\right)F^*_{21,c,120}\\
&\quad-\sum_{\substack{0<c<120\\ c\equiv 13\ (40)}}\left(\frac{c}{3}\right)F^*_{33,c,120} + \sum_{\substack{0<c<120\\ c\equiv 39\ (40)}}\left(\frac{c}{3}\right)F^*_{51,c,120} - \sum_{\substack{0<c<120\\ c\equiv 37\ (40)}}\left(\frac{c}{3}\right)F^*_{57,c,120}\\
&\quad-\sum_{\substack{0<c<120\\ c\equiv 29\ (40)}}\left(\frac{c}{3}\right)F^*_{81,c,120} + \sum_{\substack{0<c<120\\ c\equiv 33\ (40)}}\left(\frac{c}{3}\right)F^*_{93,c,120}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}q^{-\frac{63}{40}}\Bigg(\sum_{\substack{0<c<120\\ c\equiv 9\ (40)}}\left(\frac{c}{3}\right)F^*_{21,c,120} - \sum_{\substack{0<c<120\\ c\equiv 21\ (40)}}\left(\frac{c}{3}\right)F^*_{9,c,120}\\
&\qquad + \sum_{\substack{0<c<120\\ c\equiv 1\ (40)}}\left(\frac{c}{3}\right)F^*_{69,c,120} - \sum_{\substack{0<c<120\\ c\equiv 29\ (40)}}\left(\frac{c}{3}\right)F^*_{81,c,120}\\
&\qquad + \sum_{\substack{0<c<120\\ c\equiv 17\ (40)}}\left(\frac{c}{3}\right)F^*_{117,c,120} - \sum_{\substack{0<c<120\\ c\equiv 37\ (40)}}\left(\frac{c}{3}\right)F^*_{57,c,120}\\
&\qquad + \sum_{\substack{0<c<120\\ c\equiv 33\ (40)}}\left(\frac{c}{3}\right)F^*_{93,c,120} - \sum_{\substack{0<c<120\\ c\equiv 13\ (40)}}\left(\frac{c}{3}\right)F^*_{33,c,120}\Bigg)\\
&= \frac{\sqrt{3}}{480\pi}q^{-\frac{63}{40}}\sum_{b=0}^{119}\left(\frac{b}{3}\right)\big(\zeta_{120}^{111b}E_{21,b,120} - \zeta_{120}^{99b}E_{9,b,120}\\
&\qquad +\zeta_{120}^{39b}E_{69,b,120} - \zeta_{120}^{51b}E_{81,b,120}\\
&\qquad +\zeta_{120}^{63b}E_{117,b,120} - \zeta_{120}^{3b}E_{57,b,120}\\
&\qquad +\zeta_{120}^{87b}E_{93,b,120} - \zeta_{120}^{27b}E_{33,b,120}\big).
\end{aligned}
$$

This leads us to define

$$
\begin{aligned}
H(\tau) := \Bigg(\sum_{b=0}^{119}\left(\frac{b}{3}\right)\big(&\zeta_{120}^{111b}E_{21,b,120} - \zeta_{120}^{51b}E_{81,b,120} + \zeta_{120}^{39b}E_{69,b,120} - \zeta_{120}^{99b}E_{9,b,120}\\
&+\zeta_{120}^{63b}E_{117,b,120} - \zeta_{120}^{3b}E_{57,b,120} + \zeta_{120}^{87b}E_{93,b,120} - \zeta_{120}^{27b}E_{33,b,120}\big)\Bigg)^{20}.
\end{aligned}
$$

6 Identities Under Transformation by the Congruence Subgroup

For all our identities we have already shown that the LHS (as a theta function) raised to some even power is modular for Γ', where

$$\Gamma' = \Gamma_0\left(\frac{N}{2}\right)\cap\Gamma^0(2),$$

further intersected with $\Gamma_1(10)$ for the heptagonal case. As we have also seen, the RHS of each identity is an Eisenstein series, and is therefore modular with respect to $\Gamma(N)$, where thankfully the N of the LHS is equal to the N of the RHS. In order to reduce the amount of coefficients needed to be calculated, we would like to show that the RHS (raised to the same even power as the LHS), is also modular for Γ'. As the RHS is already modular for $\Gamma(N)$, we only need to consider elements in $\Gamma'' = \Gamma'/\Gamma(N)$. A set of generators for Γ'' is

$$A = \begin{pmatrix} 1 & 2\\ 0 & 1\end{pmatrix},\quad B = \begin{pmatrix} 1 & 0\\ \frac{N}{2} & 1\end{pmatrix},\quad C_0 = -I,\quad C_1 = \begin{pmatrix} c_1 & 0\\ 0 & c_1^{-1}\end{pmatrix},\quad C_2 = \begin{pmatrix} c_2 & 0\\ 0 & c_2^{-1}\end{pmatrix},\quad \dots$$

where the C_j continue to include all c_j necessary to generate $(\mathbb{Z}/N\mathbb{Z})^*$ (noticing we have already included -1). All identities have a RHS of the form

$$H(\tau) = \left(\sum_{\beta=0}^{N-1}\left(\frac{\beta}{p}\right)\sum_{(k_i,a_i)}(-1)^{i+1}\zeta_N^{k_i\beta}E_{a_i,\beta,N}(\tau)\right)^{2v}$$

for some integer v, with the second sum running over the pairs (k_i, a_i), $i = 1, 2, \dots$. For our triangular identities, there are either 1 or 2 pairs, for our pentagonal, 2 or 4 pairs. For the single heptagonal identity, there are 8 pairs.

6.1 Invariance Under Transformation by Matrix A

We begin this section with the following lemma.

Lemma 6.1. *Let p be an odd prime, $p \mid N$. Let $\gcd(d,N) = g$, and $kg = d$, $Mg = N$, all integers. Then*

$$\sum_{b=0}^{N-1} \left(\frac{b}{p}\right) \zeta_N^{-db} = \sum_{b=0}^{N-1} \left(\frac{b}{p}\right) \zeta_M^{-kb} = \begin{cases} 0 & p \neq M \\ \left(\frac{k}{p}\right) \frac{N}{p}\sqrt{p} & p = M \equiv 1 \ (mod\ 4) \\ -\left(\frac{k}{p}\right) \frac{iN}{p}\sqrt{p} & p = M \equiv 3 \ (mod\ 4). \end{cases}$$

Proof.

$$\begin{aligned} S &= \sum_{b=0}^{p-1} \left(\frac{b}{p}\right) \sum_{r=0}^{\frac{N}{p}-1} \zeta_M^{-k(b+pr)} \\ &= \sum_{b=0}^{p-1} \left(\frac{b}{p}\right) \zeta_M^{-kb} \sum_{r=0}^{\frac{N}{p}-1} \zeta_M^{-kpr} \\ &= \sum_{b=0}^{p-1} \left(\frac{b}{p}\right) \zeta_M^{-kb} \sum_{r=0}^{\frac{N}{p}-1} \zeta_{\frac{N}{p}}^{-kgr}. \end{aligned}$$

Now,

$$\sum_{r=0}^{\frac{N}{p}-1} \zeta_{\frac{N}{p}}^{-kgr} = \begin{cases} 0 & kg \not\equiv 0 \ (\mathrm{mod}\ \frac{N}{p}) \\ \frac{N}{p} & kg \equiv 0 \ (\mathrm{mod}\ \frac{N}{p}). \end{cases}$$

Clearly, $\gcd(k, M) = 1$. Hence for some $t \in \mathbb{Z}$,

$$kg \equiv 0 \ (\mathrm{mod}\ \frac{N}{p}) \iff k\frac{N}{M} = \frac{N}{p}t \iff kp = Mt \iff M \mid p.$$

So $S = 0$ when $p \not\equiv 0 \ (\mathrm{mod}\ M)$, and as p an odd prime, $p \neq M$. When $p \equiv 0 \ (\mathrm{mod}\ M)$, i.e. $p = M$, we have

$$S = \frac{N}{p} \sum_{b=0}^{p-1} \left(\frac{b}{p}\right) \zeta_p^{-kb} = \begin{cases} \left(\frac{k}{p}\right) \frac{N}{p}\sqrt{p} & p \equiv 1 \ (\mathrm{mod}\ 4) \\ -\left(\frac{k}{p}\right) \frac{N}{p} i\sqrt{p} & p \equiv 3 \ (\mathrm{mod}\ 4), \end{cases}$$

which completes the proof. □

To show that $H(\tau)$ is invariant under the matrix A, we recall that the $E_{a,b,N}$ is an Eisenstein series, and make use of the transformation formula noted earlier. We also write

$$\sum_{j \geq 0} s_j(\alpha, \beta, N) q^{\frac{j}{N}} := E_{\alpha,\beta,N}(\tau)$$

so that $s_j(\alpha, \beta, N)$ is the $q^{\frac{j}{N}}$ coefficient of $E_{\alpha,\beta,N}(\tau)$. We also spot

$$E_{a,b+a,N}(\tau) = E_{a,b,N}(\tau+1) = 2\pi i \left(\frac{a}{N} - \frac{1}{2} - \sum_{r=1}^{\infty} q^{\frac{r}{N}} \zeta_N^{r} \left[\sum_{\substack{s \mid r \\ s \equiv a\ (N) \\ s > 0}} \zeta_N^{\frac{br}{s}} - \sum_{\substack{s \mid r \\ s \equiv -a\ (N) \\ s > 0}} \zeta_N^{-\frac{br}{s}} \right] \right)$$

$$= \sum_{j \in \mathbb{N}} s_j(\alpha, \beta, N) \zeta_N^j q^{\frac{j}{N}}.$$

Thus

$$H|_A = \left(\sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \zeta_N^{k_i\beta} E_{a_i,\beta+2a_i,N} \right)^{2v}$$

$$= \left(\sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \zeta_N^{k_i\beta} \sum_{j \geq 0} s_j(a_i, \beta, N) \zeta_N^{2j} q^{\frac{j}{N}} \right)^{2v}$$

$$= \left(\sum_{j \geq 0} \zeta_N^{2j} q^{\frac{j}{N}} \sum_{(k_i,a_i)} (-1)^{i+1} \sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} s_j(a_i, \beta, N) \right)^{2v}.$$

By Lemma 6.1, we have

$$\sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} s_j(a_i, \beta, N) = 0$$

unless $s_j(a_i, \beta, N)$ has a $\zeta_N^{m\beta}$ term with $(m + k_i, N) = \frac{N}{p} := \gamma$, hence we need,

$$m = -k_i + d_i\gamma$$

with $d_i \not\equiv 0$ modulo p. As, for $j > 0$,

$$s_j(a_i, \beta, N) = -2\pi i \Bigg[\sum_{\substack{s|j \\ s \equiv a_i \ (N) \\ s>0}} \zeta_N^{\frac{\beta j}{s}} - \sum_{\substack{s|j \\ s \equiv -a_i \ (N) \\ s>0}} \zeta_N^{-\frac{\beta j}{s}} \Bigg].$$

We have $j = \pm(a_i + lN)m$ for some $l \in \mathbb{Z}$, hence $j \equiv \pm a_i m \pmod N$. We note that for all our identities $p \mid a_i$, hence $j \equiv \pm a_i m \equiv \pm a_i(-k_i + d_i\gamma) \equiv \pm a_i k_i \pmod N$.

For triangular identities of the form that have just one pair of (a_i, k_i), we notice that $a_i k_i \equiv \frac{N}{4}$ modulo N. Hence $\zeta_N^{2j} = \pm 1$ for all j of non vanishing terms. These identities were raised to the power of 2, hence $H|_A = H$.

For triangular identities that have two pairs of (a_i, k_i), we see that $a_i k_i \equiv \frac{N}{8}$ or $\pm\frac{3N}{8}$ modulo N. Hence $\zeta_N^{2j} = \pm i$ for all j of non vanishing terms. These identities were raised to the power of 4, hence $H|_A = H$.

For pentagonal identities that have two pairs of (a_i, k_i), we see that $a_i k_i \equiv \frac{3N}{4}$ (identity (17)) or $\frac{5N}{12}$ (identities (20, 23)) modulo N. Hence for identity (17) $\zeta_N^{2j} = -1$, and for the others, $\zeta_N^{2j} = \zeta_6^5$ for all j of non vanishing terms. Identity (17) was raised to the power of 2, and identities (20) and (23) were raised to the power of 6, hence in both cases $H|_A = H$.

For pentagonal identities that have four pairs of (a_i, k_i), we see that $a_i k_i \equiv \frac{7N}{8}$ or $\frac{3N}{8}$ (identities (18,19)) or $\frac{N}{24}$, $\frac{11N}{24}$, $\frac{13N}{24}$, $\frac{17N}{24}$ (identities (21, 22, 24, 25)) modulo N. Hence for identity (18, 19) $\zeta_N^{2j} = -i$, and for the others, $\zeta_N^{2j} = \zeta_{12}$, ζ_{12}^{11}, or ζ_{12}^5 for all j of non vanishing terms. All of these identities were raised to the power of 12, hence in both cases $H|_A = H$.

For the one heptagonal identity we have $a_i k_i \equiv 51 \equiv \frac{17N}{40}$ for all 8 pairs. Thus $\zeta_N^{2j} = \zeta_{20}^{17}$, and as the identity is raised to the power of 20, $H|_A = H$.

6.2 Invariance Under Transformation by Matrix B

To show that $H(\tau)$ is invariant under the matrix $B = \begin{pmatrix} 1 & 0 \\ \frac{N}{2} & 1 \end{pmatrix}$ we notice

$$H|_B = H|_{B_1^{-1}}|_{B_2}|_{B_1}$$

with

$$B_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & -\frac{N}{2} \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

because $B = B_1^{-1} B_2 B_1$. If $H|_{B_1}$ remains fixed under transformation by B_2, then H will be fixed by B. Therefore we just need to check that $H|_{B_1}$ is a sum of powers of $q^{\frac{2}{N}}$.

$$H|_{B_1} = \left(\sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \zeta_N^{k_i\beta} E_{\beta,-a_i,N} \right)^{2v}$$

$$= \left(2\pi i \sum_{(k_i,a_i)} (-1)^{i+1} \sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} \left[\frac{\beta}{N} - \frac{1}{2} - \sum_{r=1}^{\infty} q^{\frac{r}{N}} \left[\sum_{\substack{s|r \\ s\equiv\beta\ (N) \\ s>0}} \zeta_N^{-\frac{a_i r}{s}} - \sum_{\substack{s|r \\ s\equiv-\beta\ (N) \\ s>0}} \zeta_N^{\frac{a_i r}{s}} \right] \right] \right)^{2v}$$

$$= \left(S - 2\pi i \sum_{(k_i,a_i)} (-1)^{i+1} \sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} \left[\sum_{r=1}^{\infty} q^{\frac{r}{N}} \left[\sum_{\substack{s|r \\ s\equiv\beta\ (N) \\ s>0}} \zeta_N^{-\frac{a_i r}{s}} - \sum_{\substack{s|r \\ s\equiv-\beta\ (N) \\ s>0}} \zeta_N^{\frac{a_i r}{s}} \right] \right] \right)^{2v}$$

with S some constant. Rearranging the RHS, we get

$$H|_{B_1} = \left(S - 2\pi i \sum_{(k_i,a_i)} (-1)^{i+1} \sum_{r=1}^{\infty} q^{\frac{r}{N}} \left[\sum_{\beta=0}^{N-1} \sum_{\substack{s|r \\ s\equiv\beta\ (N) \\ s>0}} \left[\left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} \zeta_N^{-\frac{a_i r}{s}} \right] - \sum_{\beta=0}^{N-1} \sum_{\substack{s|r \\ s\equiv-\beta\ (N) \\ s>0}} \left[\left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} \zeta_N^{\frac{a_i r}{s}} \right] \right] \right)^{2v}$$

$$= \left(S - 2\pi i \sum_{(k_i,a_i)} (-1)^{i+1} \sum_{r=1}^{\infty} q^{\frac{r}{N}} \left[\sum_{\substack{s|r \\ s>0}} \left[\left(\frac{s}{p}\right) \zeta_N^{k_i s} \zeta_N^{-\frac{a_i r}{s}} \right] - \sum_{\substack{s|r \\ s>0}} \left[\left(\frac{-s}{p}\right) \zeta_N^{-k_i s} \zeta_N^{\frac{a_i r}{s}} \right] \right] \right)^{2v}$$

$$= \left(S - 2\pi i \sum_{r=1}^{\infty} q^{\frac{r}{N}} \left[\sum_{\substack{s|r \\ s>0}} \left(\frac{s}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \left[\zeta_N^{k_i s - a_i \frac{r}{s}} - \left(\frac{-1}{p}\right) \zeta_N^{-k_i s + a_i \frac{r}{s}} \right] \right] \right)^{2v}.$$

Define

$$G := \sum_{(k_i,a_i)} (-1)^{i+1} \left[\zeta_N^{k_i s - a_i \frac{r}{s}} - \left(\frac{-1}{p}\right) \zeta_N^{-k_i s + a_i \frac{r}{s}} \right].$$

We aim to show $G = 0$ whenever r is odd. Suppose first that $p \equiv 1 \pmod 4$, then, as we always have an even number of pairs of (a_i, k_i), and using a simple trigonometric identity,

$$G = \sum_{(k_i,a_i)} (-1)^{i+1} \left[\zeta_N^{k_i s - a_i \frac{r}{s}} - \zeta_N^{-k_i s + a_i \frac{r}{s}} \right]$$

$$= 2i \sum_{(k_i,a_i)} (-1)^{i+1} \sin\left[\frac{2\pi}{N}\left(k_i s - a_i \frac{r}{s}\right)\right]$$

$$= 2i \sum_{\text{odd } i} \left(\sin\left[\frac{2\pi}{N}\left(k_i s - a_i \frac{r}{s}\right)\right] - \sin\left[\frac{2\pi}{N}\left(k_{i+1} s - a_{i+1} \frac{r}{s}\right)\right]\right)$$

$$= 4i \sum_{\text{odd } i} \cos\left[\frac{\pi}{N}\left(s(k_i + k_{i+1}) - \frac{r}{s}(a_i + a_{i+1})\right)\right] \sin\left[\frac{\pi}{N}\left(s(k_i - k_{i+1}) - \frac{r}{s}(a_i - a_{i+1})\right)\right].$$

Similarly, if $p \equiv 3 \pmod 4$ we have

$$G = \sum_{(k_i,a_i)} (-1)^{i+1} \left[\zeta_N^{k_i s - a_i \frac{r}{s}} + \zeta_N^{-k_i s + a_i \frac{r}{s}}\right]$$

$$= 2 \sum_{(k_i,a_i)} (-1)^{i+1} \cos\left[\frac{2\pi}{N}\left(k_i s - a_i \frac{r}{s}\right)\right]$$

$$= -4 \sum_{\text{odd } i} \sin\left[\frac{\pi}{N}\left(s(k_i + k_{i+1}) - \frac{r}{s}(a_i + a_{i+1})\right)\right] \sin\left[\frac{\pi}{N}\left(s(k_i - k_{i+1}) - \frac{r}{s}(a_i - a_{i+1})\right)\right].$$

For identities that have $p \equiv 1 \pmod 4$, we note that for odd i, $k_i - k_{i+1} \equiv a_i - a_{i+1} \equiv \frac{N}{2}$ modulo N. Thus G vanishes when r is odd, as s and $\frac{r}{s}$ must share the same parity.

For identities that have $p \equiv 3 \pmod 4$, we are not as restricted, we can have either (for odd i) $k_i - k_{i+1} \equiv a_i - a_{i+1} \equiv \frac{N}{2}$ or $k_i + k_{i+1} \equiv a_i + a_{i+1} \equiv \frac{N}{2}$ modulo N. For all of these identities we have this requirement, hence G vanishes for odd r, as required.

6.3 Invariance Under Transformation by Matrices C_j

To show these identities remain fixed under matrices of the form (disregarding the subscript for now) $C = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$ modulo N, we apply the transformation formula again:

$$H|_C = \left(\sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \zeta_N^{k_i \beta} E_{ca_i, c^{-1}\beta, N}\right)^{2v}.$$

We let $\beta' = c^{-1}\beta$ and find, as we must have $(c, N) = 1$ and $p \mid N$,

$$H|_C = \left(\sum_{\beta'=0}^{N-1} \left(\frac{c\beta'}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \zeta_N^{k_i c\beta'} E_{ca_i, c^{-1}\beta, N}\right)^{2v}$$

$$= \left(\frac{c}{p}\right)^{2v} \left(\sum_{\beta'=0}^{N-1} \left(\frac{\beta'}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \zeta_N^{k_i c\beta'} E_{ca_i, \beta', N}\right)^{2v}.$$

This will be equal to H if c acts on the pairs (k_i, a_i), by multiplication modulo N, by mapping each to $\pm(k_j, a_j)$, with $j = 1, 2, \ldots$ and so on. To see this we consider the effect of one mapping. We allow for each (k_i, a_i) to be mapped to either itself, its negative, a different (k_j, a_j), or the negative of that.

If (k_i, a_i) is mapped to itself or (k_j, a_j), there are no issues. If it is mapped to the negative of one of these we notice, using $E_{-a,-b,N} = -E_{a,b,N}$,

$$\begin{aligned}\sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{-k_i\beta} E_{-a_i,\beta,N} &= -\sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{-k_i\beta} E_{a_i,-\beta,N} \\ &= -\left(\frac{-1}{p}\right) \sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} E_{a_i,\beta,N} \\ &= -\left(\frac{-1}{p}\right) \sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} E_{a_i,\beta,N}.\end{aligned}$$

If $p \equiv 3 \pmod 4$ then the negatives cancel, and we are done. If however $p \equiv 1 \pmod 4$, we need to have either all the pairs (k_i, a_i) map to negatives, or for each pair that maps to a negative match to a pair (k_j, a_j) where j has a different parity than i, which will be accounted for by the $(-1)^{i+1}$ term.

For example, identity (7) has RHS

$$H = \left(\sum_{b=0}^{23} \left(\frac{b}{3}\right) \left(\zeta_{24}^{9b} E_{3,b,24} - \zeta_{24}^{21b} E_{15,b,24}\right)\right)^4.$$

A set of generators for $(\mathbb{Z}/24\mathbb{Z})^*$ is 5, 7, 23. Of course $23 \equiv -1$ is trivial, so we focus first on 5. We get $5 \cdot (k_1, a_1) = 5 \cdot (9, 3) = (21, 15) \equiv (k_2, a_2)$ modulo 24, and $5 \cdot (k_2, a_2) = 5 \cdot (21, 15) \equiv (9, 3) \equiv (k_1, a_1)$ modulo 24. Thus H transforms to

$$\left(\sum_{b=0}^{23} \left(\frac{b}{3}\right) \left(\zeta_{24}^{21b} E_{15,b,24} - \zeta_{24}^{9b} E_{3,b,24}\right)\right)^4 = H.$$

Similarly, for 7 we find $7 \cdot (k_1, a_1) \equiv -(k_1, a_1)$ modulo 24, and $7 \cdot (k_2, a_2) \equiv -(k_2, a_2)$ modulo 24. So H transforms to

$$\begin{aligned}&\left(\sum_{b=0}^{23} \left(\frac{b}{3}\right) \left(\zeta_{24}^{-9b} E_{-3,b,24} - \zeta_{24}^{-21b} E_{-15,b,24}\right)\right)^4 \\ &= \left(-\left(\frac{-1}{3}\right)\sum_{b=0}^{23} \left(\frac{b}{3}\right) \left(\zeta_{24}^{9b} E_{3,b,24} - \zeta_{24}^{21b} E_{15,b,24}\right)\right)^4 = H.\end{aligned}$$

This is shown in the tables below. Listed is the identity number, the value of N, a list of the c_i required, and the values of the pairs (k_i, a_i). The table then shows how the c_i permute the pairs. All permutations are in fact involutions, and we denote the fact that c_i swaps (k_i, a_i) to (k_j, a_j) as simply (i, j), whereas if c_i takes (k_i, a_i) to $-(k_j, a_j)$ and (k_j, a_j) to $-(k_i, a_i)$, we write $(i, -j)$. If (k_i, a_i) maps to itself, or to $-(k_i, a_i)$, we write (i), or $(-i)$, respectively. Of course, -1 just takes the pair (k_i, a_i) to $-(k_i, a_i)$.

Identities with 2 pairs of (k_i, a_i)

Id.	$\boldsymbol{N}$	$\boldsymbol{c_1, c_2, c_3}$	$\binom{k_1}{a_1}$	$\binom{k_2}{a_2}$	$\boldsymbol{c_1}$ **acts**	$\boldsymbol{c_2}$ **acts**
7	24	$5, 7, 23$	$\binom{9}{3}$	$\binom{21}{15}$	$(1, 2)$	$(1, -2)$
8			$\binom{21}{9}$	$\binom{9}{21}$	$(1, 2)$	$(1, -2)$
9	40	$3, 11, 39$	$\binom{5}{5}$	$\binom{25}{25}$	$(1, -2)$	$(1, -2)$
10			$\binom{5}{25}$	$\binom{25}{5}$	$(1, -2)$	$(1, -2)$
12	88	$3, 5, 87$	$\binom{33}{11}$	$\binom{77}{55}$	$(1, -2)$	$(1, 2)$
13			$\binom{33}{33}$	$\binom{77}{77}$	$(1, -2)$	$(1, 2)$
15	232	$3, 5, 231$	$\binom{29}{29}$	$\binom{145}{145}$	$(1, -2)$	$(1, 2)$
16			$\binom{145}{29}$	$\binom{29}{145}$	$(1, -2)$	$(1, 2)$
17	60	7,13,59	$\binom{35}{15}$	$\binom{5}{45}$	$(1, -2)$	$(1), (2)$
20	156	7,11,155	$\binom{143}{91}$	$\binom{65}{13}$	$(1, 2)$	$(-1), (-2)$
23	444	5,7,443	$\binom{407}{259}$	$\binom{185}{37}$	$(1, -2)$	$(1, 2)$

For the three identities (6, 11, 14), that have just a single pair, it is easy to find a set of generators that for example, maps the pair to its negative. For the single heptagonal identity, we need to avoid matrices with first entry congruent to ± 3 modulo 10. So, as $N = 120$ we take $c_0 = 119$, $c_1 = 11$, $c_2 = 19$, and $c_3 = 29$. As before, c_0 is immediate. We see that c_1 takes (k_1, a_1) to $-(k_4, a_4)$ and vice versa, (k_2, a_2) to $-(k_3, a_3)$ and vice versa, (k_5, a_5) to $-(k_8, a_8)$ and vice versa, and (k_6, a_6) to $-(k_7, a_7)$ and vice versa. The value 19 swaps the pair with index 1 with the negative of the pair with index 2, 3 with -4, 5 with -6, 7 with -8. Finally, 29 swaps 1 and 4, 2 and 3, 5 and 8, 6 and 7.

7 Equivalence Using Sturm's Bound

Let $f = \sum_{k \in \mathbb{Z}} \alpha_k q^k$. We define $ord(f)$ to be the smallest such n that $\alpha_n \neq 0$. We first state a simplified version of Sturm's bound [12].

Theorem 7.1. *Define $\Gamma := SL_2(\mathbb{Z})$, and let f, g be modular forms on $\Gamma' \supseteq \Gamma(N)$ of weight k, a positive integer. Suppose $ord(f - g) > \frac{k[\Gamma:\Gamma']}{12}$. Then $f = g$.*

In other words, Sturm's bound says that given two modular forms over the same congruence subgroup and of the same weight, then they are equivalent if their q-expansions agree up to the $q^{\frac{kM}{12}}$ coefficient, where $k = 2v$ is the weight, and M is the index of the congruence subgroup in Γ.

Identities with 4 pairs of (k_i, a_i)

Id.	$\boldsymbol{N}$	$\boldsymbol{c_1, c_2, c_3, c_4}$	$\binom{\boldsymbol{k_1}}{\boldsymbol{a_1}}$	$\binom{\boldsymbol{k_2}}{\boldsymbol{a_2}}$	$\binom{\boldsymbol{k_3}}{\boldsymbol{a_3}}$	$\binom{\boldsymbol{k_4}}{\boldsymbol{a_4}}$	$\boldsymbol{c_1}$ **acts**	$\boldsymbol{c_2}$ **acts**	$\boldsymbol{c_3}$ **acts**
18	120	$7, 11, 19, 119$	$\binom{105}{25}$	$\binom{45}{85}$	$\binom{105}{65}$	$\binom{45}{5}$	$(1,-3),(2,-4)$	$(1,-2),(3,-4)$	$(1,-4),(2,-3)$
19			$\binom{105}{5}$	$\binom{45}{65}$	$\binom{105}{85}$	$\binom{45}{25}$	$(1,-3),(2,-4)$	$(1,-2),(3,-4)$	$(1,-4),(2,-3)$
21	264	$5, 7, 13, 263$	$\binom{121}{11}$	$\binom{253}{143}$	$\binom{209}{187}$	$\binom{187}{209}$	$(1,-4),(2,3)$	$(1,-3),(2,4)$	$(1,2),(3,-4)$
22			$\binom{121}{121}$	$\binom{187}{187}$	$\binom{209}{209}$	$\binom{253}{253}$	$(1,-2),(3,4)$	$(1,-3),(2,4)$	$(1,4),(2,-3)$
24	696	$5, 7, 13, 695$	$\binom{493}{29}$	$\binom{145}{377}$	$\binom{319}{551}$	$\binom{667}{203}$	$(1,-3),(2,-4)$	$(1,4),(2,3)$	$(1,2),(3,4)$
25			$\binom{319}{667}$	$\binom{667}{319}$	$\binom{493}{145}$	$\binom{145}{493}$	$(1,-3),(2,-4)$	$(1,4),(2,3)$	$(1,2),(3,4)$

Recall we have defined for the triangular and pentagonal identities,

$$\Gamma' = \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2),$$

and for the heptagonal identity,

$$\Gamma' = \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2) \cap \Gamma_1(10).$$

We have now shown that each side of our transformed identities are modular of weight $2v$ for $\Gamma'' = \Gamma'/\Gamma(N)$. We can now apply Sturm's bound. We have [2, p. 14], for the triangular and pentagonal cases,

$$M = [\Gamma : \Gamma'] = [\Gamma : \Gamma_0(N)] = N \prod_{p|N}\left(1 + \frac{1}{p}\right)$$

as, despite the fact that $\Gamma' \neq \Gamma_0(N)$, they have the same index in Γ. To see this, we simply notice that Γ' and $\Gamma_0(N)$ both share a common subgroup, $\Gamma_0(N) \cap \Gamma^0(2)$, and this common subgroup clearly has index 2 in both. For the heptagonal case, we simply have to double M. We need to check $\frac{kM}{12} + 1$ coefficients. Starting with identity (6), we have $M = 36$, and we need to check $\frac{kM}{12} + 1 = 7$ coefficients. We use the form of

$$q^{\frac{\alpha+\beta}{8(k-2)}} G_k(q^\alpha) G_k(q^\beta) = \sum_{(a_i, c_i)} F^*_{a_i, c_i, 4(k-2)},$$

for ease, which as we mentioned before each identity can be written as. For identity (6), this is

$$q^{\frac{3}{4}} G_3(q) G_3(q^5) = F^*_{5,3,20} + F^*_{5,7,20}$$

and find both sides start with

$$1 + 1 \cdot q + 0 \cdot q^2 + 1 \cdot q^3 + 0 \cdot q^4 + 1 \cdot q^5 + 2 \cdot q^6 + 0 \cdot q^7.$$

As the coefficients of this form agree up to the required amount for Sturm's bound, the coefficients of $H(\tau)$ and $\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right)^{2v}$ must also agree up to the required amount. Hence by Sturm's bound $H(\tau) = \vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right)^{2v}$, and so the LHS is equivalent to the RHS up to multiplication by some constant (a root of unity). But we've seen that the coefficients agree, so this constant is 1, and the identity holds. See appendix for the code used.

For identity (7), we have $M = 48$, and so we need to check 17 coefficients. We check the coefficients of the same form

$$q^{\frac{7}{8}} G_3(q) G_3(q^6) = F^*_{3,7,24} + F^*_{9,5,24}$$

and find both sides start with

$$1+1\cdot q+0\cdot q^2+1\cdot q^3+0\cdot q^4+0\cdot q^5+2\cdot q^6+1\cdot q^7+0\cdot q^8+1\cdot q^9+1\cdot q^{10}+0\cdot q^{11}+1\cdot q^{12}+0\cdot q^{13}+0\cdot q^{14}+1\cdot q^{15}+1\cdot q^{16}+0\cdot q^{17}.$$

As before, this is enough to show identity (7) holds.

Our computations show that both the LHS and RHS have the same coefficients for all triangular, pentagonal, and heptagonal identities. Notice that while we always knew both sides were modular forms for $\Gamma(N)$, using that would have made the number of coefficients required for Sturm's Bound much larger, hence the efforts to show both sides were modular forms for the larger congruence subgroups.

8 Appendix

We used a simple code in MATLAB to check the coefficients. There may very well exist a more elegant way of doing this, but as we always had $N < 1000$ this was suitable for our purposes. The code first calculates the triangular/pentagonal/heptagonal numbers up to the number required, then sums the appropriate multiples and checks the frequency to find the coefficients of the LHS. For the RHS the code uses the definition of $F_{a,c,N}$ to calculate the coefficients using the pairs (a_i, c_i). Finally, these two lists are compared, returning a value of 0 if the coefficients are the same. If the coefficients agree up to the necessary limit, then of course the coefficients agree when raised to the power $2v$.

Included below is the code used for the triangular numbers. We merely altered the first section so it would calculate the correct number of coefficients, and the correct numbers, for the pentagonal and heptagonal number identities.

8.1 Triangular Numbers Code

```
%% Script to find number of ways each number can be represented as the sum of a
%% triangular # and a multiple of a triangular #
% Numbers that cannot be represented in any such way are excluded from
% output. Output is each number with the frequency at which it can be
% expressed in the above way.

clear

% IDENTITY VALUES GO HERE
% alpha =  ; beta =  ; % values of alpha, beta, alpha < beta
% d = ; % power of q multiplied by N
% N =  ; % value of N
% M =  ; % index of subgroup
% k  = ; % weight of form
% a = [ ]; c = [ ]; % values of a, c

noF = length(c); % number of F_{a,c,N}
m = k*M/12 + 1; % number of coefficients needed
n = floor( ( -1 + sqrt( 1 + 8 * m / alpha) ) / 2 ) + 1; % number of
% triangular numbers needed
tri = zeros(n, 1);
sums = zeros(n);

for i = 0 : n
    tri(i+1) = i * ( i + 1 ) / 2;
end
atri = alpha .* tri;
btri = beta .* tri;

for i = 1 : n
    for j = 1 : n
        sums(i, j) = atri(i) + btri(j);
    end
end
```

```
out = [unique(sums),histc(sums(:),unique(sums))];
index = find(out(:, 1) < m + 1);
result = out(1 : max(index), :);

%% Working out coefficients of the RHS

Fpos = zeros(noF, m, m);
Fneg = zeros(noF, m, m);

for i = 1 : noF
    for j = 1 : m + 1
        for k = 1 : m + 1
            Fpos(i, j, k) = ( (a(i) + (k - 1) * N) * (c(i) + (j - 1) * N) - d )/ N;
            Fneg(i, j, k) = ( ( -a(i) + k * N) * ( -c(i) + j * N) - d )/ N;
        end
    end
end

outpos = [unique(Fpos),histc(Fpos(:),unique(Fpos))];
indexpos = find(outpos(:, 1) < m + 1);
resultpos = outpos(1 : max(indexpos), :);

outneg = [unique(Fneg),histc(Fneg(:),unique(Fneg))];
indexneg = find(outneg(:, 1) < m + 1);
resultneg = outneg(1 : max(indexneg), :);

resultF = zeros(m, 2);
k = max(length(resultpos), length(resultneg));
for i = 1 : k
    if isempty(find(resultneg(:, 1) == resultpos(i, 1), 1))
        resultF(i, :) = resultpos(i, :);
    else
        resultF(i, 1) = resultpos(i, 1);
        resultF(i, 2) = resultpos(i, 2) - resultneg(find(resultneg(:, 1)
        == resultpos(i, 1), 1), 2);
        resultneg(find(resultneg(:, 1) == resultpos(i, 1), 1), :) = [];
    end
end

resultneg(:, 2) = -resultneg(:, 2);
resultF = [resultF; resultneg];
resultF = resultF(any(resultF, 2), :);
resultF = resultF(any(resultF(:, 2), 2), :);

%% Comparing

any(any(result-resultF)) % Checking if the difference between the two results
% is zero. If zero, it will return zero.
```

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